Explicit Formulas for Noncommutative $\mathbf{C}P^N$ and $\mathbf{C}H^N$

Hiroshi Umetsu with A. Sako and T. Suzuki

Kushiro National College of Technology

based on J. Math. Phys. **53**, 073502 (2012) [arXiv:1204.4030]

Sep. 14, 2012, Belgrade, Serbia

1. Introduction

Noncommutative spaces in physics

- A charged particle in a strong constant magnetic field (B_z)

$$L \sim B_z \left(\dot{x}(t)y(t) - \dot{y}(t)x(t)\right) \quad \Longrightarrow \quad [x(t),\ y(t)] \sim i\hbar/B_z$$

- String theories

  Low energy effective theories on D-branes in a constant background $NS - NS$ B field.

$$B_{\mu\nu} \ \sim \ \text{noncommutative parameter}$$

- Matrix models

  noncommutative coordinates:

$$[x_\mu, x_\nu] \neq 0 \qquad \Longrightarrow \quad N \times N \text{ matrices } \hat{X}_\mu$$

Properties of field theories on N.C. spaces

- UV/IR mixing and nonlocality

  ▷ Low energy (IR) scales appear in high energy (UV) phenomena (e.g. UV divergences).

  Ex. ϕ^4 theory on N.C. $\mathbb{R}^4$

  one-loop self-energy:

$$\Gamma^{(2)}(p) = p^2 + M^2 + \frac{cg}{(\theta^2 p^2 + 1/\Lambda^2)^2} + \cdots$$

  ▷ Open Wilson lines become observables in N.C. Yang-Mills theories.

$$W[C_{12}; k] = \int d^4 x_1 \mathrm{Tr} \left[P e^{i \int_{C_{12}} A} \right]_* * e^{ik \cdot x_2}$$

$$\begin{array}{rcl} \text{Length of the open Wilson line} & \sim & \text{energy scale} \\ x^1 - x^2 & \sim & k\theta \end{array}$$

- N.C. soliton and Instanton

Nonperturbative effects characteristic in N.C. field theories

Ex. N.C. solitons in a scalar field theory on $\mathbb{R}^4$

$$\mathcal{L} = \frac{1}{2\theta}\partial_\mu \phi \partial^\mu \phi + V(\phi)$$

In the large θ limit, the equation of motion becomes

$$\frac{\partial V(\phi)}{\partial \phi} = 0$$

Nontrivial solutions (N.C. solitons) can be constructed based on the projection operator,

$$\phi \sim \lambda P, \qquad (P * P = P,\ \lambda : \text{constant})$$

More models of tractable field theories on N.C. spaces are needed to investigate further these properties.

Construction of N.C. space

- Moyal product on $\mathbb{R}^d$

$$(f * g)(x) = e^{\frac{i}{2}\theta^{ij}\partial_i^x \partial_j^y} f(x)g(y)\Big|_{y=x}, \qquad [x^i,\ x^j]_* = i\theta^{ij}$$

- Fuzzy spaces from matrix models

  Ex. Fuzzy S^2

$$[\hat{X}_i,\ \hat{X}_j] = i\epsilon_{ijk}\hat{X}_k,$$

$$\hat{X}_i\ (i = 1, 2, 3) :\ (2L+1) \times (2L+1)\text{matrices}$$

  fields on N.C. space:

$$\phi(x) \quad \Longrightarrow \quad (2L+1) \times (2L+1) \text{ matrix } \hat{\phi}(\hat{X})$$

  action: $S[\hat{X}] = \mathrm{Tr}\ L(\hat{\phi})$

- Geometric quantization

  A generalization of the canonical quantization

Deformation Quantization (weak sense)

Deformation Quantization is defined as follows. $\mathcal{F}$ is defined as a set of formal power series of $\hbar$:

$$\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k \right\}.$$

A star product is defined as

$$f * g = \sum C_k(f,g)\hbar^k$$

s.t. the product satisfies the following conditions.

1. $*$ is associative product.
2. C_k is a bidifferential operator.
3. $C_0(f,g) = fg, \qquad C_1(f,g) - C_1(g,f) = i\{f,g\}.$
4. $f * 1 = 1 * f.$

2. Star product with separation of variables

Kähler manifold with a Kähler potential Φ and a Kähler 2-form ω

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \qquad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi,$$

where $\partial_i = \partial / \partial z^i,\ \partial_{\bar{j}} = \partial / \partial z^{\bar{j}}$.

$*$ is called a star product with separation of variables when

$$a * f = af$$

for holomorphic function a and

$$f * b = fb$$

for anti-holomorphic function b.

- A. V. Karabegov showed that for arbitrary ω, there exists a star product with separation of variables $*$.

A. V. Karabegov, Commun. Math. Phys. **180**, 745 (1996)

In this method for making deformation quantization, a star product is constructed as a formal series of differential operators.

star product		differential operator
$f * g$	$=$	$L_f\ g$: left $*$ multiplication by f
$g * f$	$=$	$R_f\ g$: right $*$ multiplication by f

L_f (R_f) is a differential operator corresponding to a left (right) $*$ multiplication by f:

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n,$$

where

$$A_n = a_{n,\alpha}(f) \prod_i \left(D^{\bar{i}}\right)^{\alpha_i}, \qquad (D^{\bar{i}} = g^{\bar{i}j}\partial_j).$$

D's satisfy the following relations:

$$\left[D^{\bar{i}},\ D^{\bar{j}}\right] = 0, \qquad \left[D^{\bar{i}},\ \partial_{\bar{j}}\Phi\right] = \delta^{i}_{j}$$

It is required that L_f satisfies

$$L_f 1 = f * 1 = f,$$
$$L_f\left(L_g h\right) = f * (g * h) = (f * g) * h = L_{L_f g} h.$$

L_f which has these properties is determined by the following conditions,

$$[L_f,\ \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0,$$

and $A_0 = f$.

Note:

Because of $R_{\partial_{\bar{i}}\Phi} = \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}$, this condition is equivalent to

$$[L_f,\ R_{\partial_{\bar{i}}\Phi}]\ g = f * (g * \partial_{\bar{i}}\Phi) - (f * g) * \partial_{\bar{i}}\Phi = 0.$$

This condition is equivalent to the recursion relations,

$$[A_n,\ \partial_{\bar{i}}\Phi] = [\partial_{\bar{i}},\ A_{n-1}],$$

at each order of $\hbar$.

If one obtains the operator $L_{\bar{z}^i}$,

$$L_{\bar{z}^i} g = \bar{z}^i * g,$$

L_f coorrsponding to an arbitrary function f is given by

$$L_f = \sum_\alpha \frac{1}{\alpha} \left(\frac{\partial}{\partial \bar{z}} \right)^\alpha f \left(L_{\bar{z}} - \bar{z} \right)^\alpha .$$

where α is a multi-index, $\alpha = (\alpha_1, \cdots, \alpha_m)$.

It is not easy to derive explicit expressions of star products in all order of $\hbar$ by solving the recursion relations.

3. N.C. deformation of $\mathbb{C}P^N$

Inhomogeneous coordinates z^i $(i = 1, 2, \cdots, N)$

Kähler potential of $\mathbb{C}P^N$:

$$\Phi = \ln\left(1 + |z|^2\right), \qquad (|z|^2 = \sum_i z^i \bar{z}^i)$$

Metric $(g_{i\bar{j}})$: $ds^2 = 2g_{i\bar{j}} dz^i d\bar{z}^j$,

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}$$

The following relations simplify our calculations for L_f in the case of $\mathbb{C}P^N$,

$$\partial_{\bar{i}_1} \partial_{\bar{i}_2} \cdots \partial_{\bar{i}_n} \Phi = (-1)^{n-1}(n-1)!\, \partial_{\bar{i}_1} \Phi \partial_{\bar{i}_2} \Phi \cdots \partial_{\bar{i}_n} \Phi,$$

Riemann tensor: $R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}} g_{k\bar{l}} - g_{i\bar{l}} g_{k\bar{j}}$.

Construction of $L_{\bar{z}^l}$ $\quad (L_{\bar{z}^l} f = \bar{z}^l * f)$

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n A_n,$$

where A_n $(n \geq 2)$ is a formal series of $D^{\bar{k}}$.

- We assume that A_n has the following form,

$$A_n = \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}.$$

From $[L_{\bar{z}^l}, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0$, A_n are recursively determined by

$$[A_n, \partial_{\bar{i}} \Phi] = [\partial_{\bar{i}}, A_{n-1}], \qquad (n \geq 2)$$

where $A_1 = D^{\bar{l}}$.

- After some calculations, we found the following recursion relation

$$a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1)a_m^{(n-1)}.$$

and $a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1$.

- To solve these equations, we introduce a generating function

$$\alpha_m(t) \equiv \sum_{n=m}^{\infty} t^n a_m^{(n)}, \qquad (m \geq 2).$$

From the recursion relation, $\alpha_m(t)$ is determined as

$$\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1-m+\frac{1}{t})}{\Gamma(1+\frac{1}{t})}, \qquad (m \geq 2).$$

The coefficient $a_m^{(n)}$ is related to the Stirling number of the second kind $S(n,k)$,

$$a_m^{(n)} = S(n-1, m-1).$$

- Summarizing the above calculations, $L_{\bar{z}^l}$ becomes

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}$$

$$= \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\hbar) \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}.$$

Similarly, the operator R_{z^i} corresponding to the right $*$ multiplication of z^i is obtained.

- Star products among z^i and $\bar{z}^i$,

$$z^i * z^j = z^i z^j, \qquad z^i * \bar{z}^j = z^i \bar{z}^j, \qquad \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j,$$

$$\bar{z}^i * z^j = \bar{z}^i z^j + \hbar \delta_{ij} (1 + |z|^2) {}_2F_1 \left(1, 1; 1 - 1/\hbar; -|z|^2\right)$$
$$+ \frac{\hbar}{1-\hbar} \bar{z}^i z^j (1 + |z|^2) {}_2F_1 \left(1, 2; 2 - 1/\hbar; -|z|^2\right).$$

- L_f for an arbitrary function f

$$L_f = \sum_\alpha \frac{1}{\alpha} \left(\frac{\partial}{\partial \bar{z}} \right)^\alpha f \left(L_{\bar{z}} - \bar{z} \right)^\alpha$$

We can derive an explicit formula for L_f,

$$L_f = \sum_{n=0}^{\infty} \frac{\alpha_n(\hbar)}{n!} g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f \right) D^{\bar{k}_1} \cdots D^{\bar{k}_n}.$$

It is shown that L_f satisfies $[L_f, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0$.

This star product on $\mathbb{C}P^N$ is characterized by a single function of $\hbar$, $\alpha_n(\hbar)$.

- Leibniz rule for differentials

In general,

$$\partial(f * g) \neq (\partial f) * g + f * (\partial g).$$

The Killing vectors corresponding to the $SU(N+1)$ isometry of $\mathbb{C}P^N$

$$\mathcal{L}_a = \xi_a^i \partial_i + \xi_a^{\bar{i}} \partial_{\bar{i}}, \qquad (a = 1, 2, \cdots, (n+1)^2 - 1),$$

$$[\mathcal{L}_a,\ \mathcal{L}_b] = i f_{abc} \mathcal{L}_c, \quad (f_{abc} : \text{ structure constant of } SU(N+1))$$

The Leibniz rule holds with respective to the Killing vectors,

$$\mathcal{L}_a(f * g) = (\mathcal{L}_a f) * g + f * (\mathcal{L}_a g).$$

This property is important to construct actions of field theories on the N.C. $\mathbb{C}P^N$ which is invariant under the isometry.

Comparison with other N.C. deformations for $\mathbb{C}P^N$

1. Bordemann, Brischle, Emmrich and Waldmann gave a star product on $\mathbb{C}P^N$ by performing the phase space reduction from $\mathbb{C}^{N+1}\backslash\{0\}$.

   Bordemann, et al, Lett. Math. Phys. **36** (1996), 357

$$f *_B g = fg + \sum_{m=1}^{\infty} \hbar^m \sum_{s=1}^{m} \sum_{k=1}^{s} \frac{k^{m-1}(-1)^{m-k}}{s!(s-k)!(k-1)!} \left(|\zeta|^2\right)^s \times \frac{\partial^s f}{\partial \bar{\zeta}^{A_1} \dots \bar{\zeta}^{A_s}} \frac{\partial^s g}{\partial \zeta^{A_1} \dots \zeta^{A_s}},$$

   where $\zeta^{A_i}, \bar{\zeta}^{A_j}$ are the homogeneous coordinates of $\mathbb{C}P^N$.

   We showed that this star product $*_B$ coincides with the one we obtained,

$$f *_B g = f * g.$$

2. Balachandran, Dolan, Lee, Martin and O'Connor derived an star product on a fuzzy $\mathbb{C}P^N$ by using matrix representations of $SU(N+1)$. Balachandran, et al., J. Geom. Phys. **43**, 184 (2002)

Their star product also coincides with the one we derived, if one considers the specific case

- $\hbar = 1/L$ ($L \in \mathbb{N}$: matrix size)
- star product in a function space spanned by

$$\frac{z^{i_1} \dots z^{i_m} \bar{z}^{j_1} \dots \bar{z}^{j_n}}{(1+|z|^2)^L}, \qquad (m, n \leq L).$$

Note: L_f can be rewritten by the use of the covariant derivatives on $\mathbb{C}P^N$ as

$$L_f = \sum_{n=0}^{\infty} \frac{\alpha_n(\hbar)}{n!} g^{\bar{j}_1 k_1} \cdots g^{\bar{j}_n k_n} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f\right) \nabla_{k_1} \cdots \nabla_{k_n}.$$

As far as we know, the origin of this coincidence of the star products obtained by these different methods is not apparent at this time.

4. Fock representation

$\{z^i, \partial_j \Phi \mid i,j = 1,2,\cdots,N\}$ and $\{\bar{z}^i, \partial_{\bar{j}} \Phi \mid i,j = 1,2,\cdots,N\}$ constitute $2N$ sets of the creation-annihilation operators under the star product,

$$\left[\partial_i \Phi,\ z^j\right]_* = \hbar \delta_{ij}, \qquad \left[z^i,\ z^j\right]_* = 0, \qquad \left[\partial_i \Phi,\ \partial_j \Phi\right]_* = 0,$$
$$\left[\bar{z}^i,\ \partial_{\bar{j}} \Phi\right]_* = \hbar \delta_{ij}, \qquad \left[\bar{z}^i,\ \bar{z}^j\right]_* = 0, \qquad \left[\partial_{\bar{i}} \Phi,\ \partial_{\bar{j}} \Phi\right]_* = 0.$$

Annihilation operators: $\partial_i \Phi$, $\bar{z}^j$
Creation operators: z^i, $\partial_{\bar{j}} \Phi$

- $e^{-\Phi/\hbar} = (1 + |z|^2)^{-1/\hbar}$ is the vacuum projection:

$$\partial_i \Phi * e^{-\Phi/\hbar} = \bar{z}^j * e^{-\Phi/\hbar} = 0, \quad e^{-\Phi/\hbar} * z^i = e^{-\Phi/\hbar} * \partial_{\bar{j}} \Phi = 0,$$
$$e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar}.$$

- A class of functions is constructed by acting the creation-annihilation operators on the vacuum projection:

$$M_{i_1\cdots i_m;j_1\cdots j_n} := c_{mn} z^{i_1} * \cdots * z^{i_m} * e^{-\Phi/\hbar} * \bar{z}^{j_1} * \cdots * \bar{z}^{j_n}$$
$$= c_{mn} z^{i_1} \cdots z^{i_m} \bar{z}^{j_1} \cdots \bar{z}^{j_n} / (1 + |z|^2)^{1/\hbar},$$

where we choose $c_{mn} = 1/\sqrt{m!n!\alpha_m(\hbar)\alpha_n(\hbar)}$.

- These functions form a closed algebra:

$$M_{i_1\cdots i_m;j_1\cdots j_n} * M_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr}\delta^{k_1\cdots k_n}_{j_1\cdots j_n} M_{i_1\cdots i_m;l_1\cdots l_s},$$

$$\delta^{k_1\cdots k_n}_{j_1\cdots j_n} = \frac{1}{n!}\left[\delta^{k_1}_{j_1}\cdots\delta^{k_n}_{j_n} + \text{permutations of } (j_1,\cdots,j_n)\right].$$

- Projection operators

$$P_{i_1\cdots i_n} = M_{i_1\cdots i_n;i_1\cdots i_n}$$

$$P_{i_1\cdots i_m} * P_{j_1\cdots j_n} = \delta_{mn}\delta^{i_1\cdots i_n}_{j_1\cdots j_n} P_{i_1\cdots i_n}$$

$$z^k * M_{i_1\cdots i_m;j_1\cdots j_n} = \sqrt{\frac{m+1}{-m+1/\hbar}} M_{ki_1\cdots i_m;j_1\cdots j_n},$$

$$\partial_{\bar{k}}\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{(m+1)(-m+1/\hbar)} M_{ki_1\cdots i_m;j_1\cdots j_n},$$

$$\partial_k\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{\frac{-m+1+1/\hbar}{m}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots \hat{i}_l \cdots i_m;j_1\cdots j_n},$$

$$\bar{z}^k * M_{i_1\cdots i_m;j_1\cdots j_n} = \frac{1}{\sqrt{m(-m+1+1/\hbar)}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots \hat{i}_l \cdots i_m;j_1\cdots j_n},$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * z^k = \frac{1}{\sqrt{n(-n+1+1/\hbar)}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots \hat{j}_l \cdots j_n},$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_{\bar{k}}\Phi = \hbar\sqrt{\frac{-n+1+1/\hbar}{n}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots \hat{j}_l \cdots j_n}$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_k\Phi = \hbar\sqrt{(n+1)(-n+1/\hbar)} M_{i_1\cdots i_m;j_1\cdots j_n k},$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \bar{z}^k = \sqrt{\frac{n+1}{-n+1/\hbar}} M_{i_1\cdots i_m;j_1\cdots j_n k}.$$

5. Explicit formulas for a star product on $\mathbb{C}H^N$

- Similarly, explicit expressions of star products on complex hyperbolic spaces $\mathbb{C}H^N$ are derived by using the deformation quantization with separation of variables.

  $\mathbb{C}H^N$: noncompact Kähler manifold

  Kähler potential: $\Phi = -\log(1 - |z|^2)$

  metric: $ds^2 = 2\frac{(1-|z|^2)\delta_{ij} + \bar{z}^i z^j}{(1-|z|^2)^2} dz^i dz^{\bar{j}}$

- star product:

$$f * g = \sum_{n=0}^{\infty} \frac{\beta_n(\hbar)}{n!} g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) \left(D^{\bar{k}_1} \cdots D^{\bar{k}_n} g\right),$$

where $\beta_n(\hbar) = -\alpha_n(-\hbar) = \frac{(-1)^{n-1}\Gamma(1/\hbar)}{\Gamma(n+1/\hbar)}$.

- Fock representation:

The following functions form a closed algebra:

$$N_{i_1\cdots i_m;j_1\cdots j_n} = c'_{mn} z^{i_1} \cdots z^{i_m} \bar{z}^{j_1} \cdots \bar{z}^{j_n} (1 - |z|^2)^{1/\hbar},$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * N_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr} \delta^{k_1\cdots k_n}_{j_1\cdots j_n} N_{i_1\cdots i_m;l_1\cdots l_s}.$$

6. Summary and discussion

- We obtained explicit expressions of star products on $\mathbb{C}P^N$ by using the deformation quantization with separation of variables.

$$f * g = \sum_{n=0}^{\infty} \frac{\alpha_n(\hbar)}{n!} g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f \right) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g.$$

- Fock representation

The following functions form a closed algebra under the star product,

$$M_{i_1 \cdots i_m ; j_1 \cdots j_m} \sim \frac{z^{i_1} \ldots z^{i_m} \bar{z}^{j_1} \ldots \bar{z}^{j_n}}{(1 + |z|^2)^{1/\hbar}}$$

$$M_{i_1 \cdots i_m ; j_1 \cdots j_n} * M_{k_1 \cdots k_r ; l_1 \cdots l_s} = \delta_{nr} \delta^{k_1 \cdots k_n}_{j_1 \cdots j_n} M_{i_1 \cdots i_m ; l_1 \cdots l_s}.$$

- Similarly, explicit expressions of star products on $\mathbb{C}H^N$ are derived by using the deformation quantization with separation of variables.

▷ Star products on other Kähler manifolds

Ex. a locally symmetric Kähler manifold: $\nabla_\mu R_{\nu\rho\sigma}{}^{\lambda} = 0$

Assumption:

$$L_f g = \sum_{n=0}^{\infty} T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f\right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g\right), \qquad (\nabla T = 0)$$

Condition: $[L_f, \partial_{\bar{i}}\Phi + \hbar \partial_{\bar{i}}] = 0$

$$\Big[n T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n} g_{k_n \bar{i}} - \hbar T_{n-1}^{\bar{j}_1 \cdots \bar{j}_{n-1}, k_1 \cdots k_{n-1}} \delta_{\bar{i}}^{\bar{j}_n} - \hbar \frac{n(n-1)}{2} T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_{n-2} pq} R_{\bar{i}pq}{}^{k_{n-1}} \Big] \times \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f\right) \left(\nabla_{k_1} \cdots \nabla_{k_{n-1}} g\right) = 0.$$

▷ Construction of field theories on N.C. $\mathbb{C}P^N$ (or $\mathbb{C}H^N$)

Ex. scalar field:

$$\phi = \sum \phi_{i_1 \cdots i_m ; j_1 \cdots j_n} M_{i_1 \cdots i_m ; j_1 \cdots j_n}$$

Lagrangian density: $\frac{1}{2} \mathcal{L}_a \phi \mathcal{L}_a \phi + V[\phi]$

N.C. soliton in the large noncommutativity limit ?

projection operators: $P_{i_1 \cdots i_n} = M_{i_1 \cdots i_n ; i_1 \cdots i_n}$