D-modules Representations of Finite Superconformal Algebras and Constraints on Superconformal Mechanics

Francesco Toppan

TEO, CBPF (MCTI)
Rio de Janeiro, Brazil

VII Mathematical Physics Conference
Belgrade, September 9-19, 2012

Based on:

Z. Kuznetsova and F. T., **JMP** 53, 043513 (2012), arXiv:1112.0995.

L. Baulieu and F. T., **NPB** 855, 742 (2012), arXiv:1108.4370.

S. Khodaee and F.T., arXiv:1208.3612.

+ D-module reps for global SQM (**last 10 years**)

Main results:

Existence of a critical scaling dimension for $\mathcal{N} = 4, 7, 8$ finite SCA's.

Constraints on superconformal mechanics in the Lagrangian setting.

Possible applications:

Test particles near BH horizon (Britto-Pacumio et al. hep-th/9911066,

CFT_1/AdS_2 correspondence,

Computation of BH entropy via l.w.r.'s and ladder operators.

The bosonic Conformal Mechanics is based on the $sl(2)$ algebra with D-module rep given by

$$
\begin{aligned}
H &= \partial_t, \\
D &= -(t\partial_t + \lambda), \\
K &= -(t^2\partial_t + 2\lambda t).
\end{aligned}
$$

λ is the scaling dimension.

For any λ we have

$$
\begin{aligned}
[D, H] &= H, \\
[D, K] &= -K, \\
[H, K] &= 2D.
\end{aligned}
$$

(D is the dilatation operator).

Finite SCA's from Kac's list of simple Lie superalgebras:

$$
\begin{array}{rcl}
\mathcal{N}=0 & : & sl(2),\\
\mathcal{N}=1 & : & osp(1,2),\\
\mathcal{N}=2 & : & sl(2|1),\\
\mathcal{N}=3 & : & osp(3|2),\\
\mathcal{N}=4 & : & A(1,1), \quad \underline{D(2,1;\alpha)},\\
\mathcal{N}=5 & : & B(2,1),\\
\mathcal{N}=6 & : & A(2,1), \quad B(1,2), \quad D(3,1),\\
\mathcal{N}=7 & : & B(3,1), \quad \underline{G(3)},\\
\mathcal{N}=8 & : & D(4,1), \quad D(2,2), \quad A(3,1), \quad \underline{F(4)},\\
\ldots & : & \ldots,
\end{array}
$$

Properties of finite SCA's:

Even sector $\mathcal{G}_{even}$: $sl(2) \oplus R$ (R is the R-symmetry).

Odd sector $\mathcal{G}_{odd}$: $2\mathcal{N}$ generators.

The dilatation operator D induces the grading

$$\mathcal{G} \quad = \quad \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1 .$$

The sector $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) containes a unique generator given by H (K).
The $\mathcal{G}_0$ sector is given by the union of D and the R-symmetry subalgebra ($\mathcal{G}_0 = \{D\} \bigcup \{R\}$).
The odd sectors $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ are spanned by the supercharges Q_i's and their superconformal partners $\widetilde{Q}_i$'s, respectively.
The invariance under the global supercharges Q_i's and the generator K implies the invariance under the full superconformal algebra $\mathcal{G}$.

Comment:

The $\mathcal{N}$-extended global supersymmetry algebra

$$
\begin{array}{rcll}
\{Q_i, Q_j\} & = & \delta_{ij} H, & \quad i,j = 1,\ldots,\mathcal{N}, \\
[H, Q_i] & = & 0, &
\end{array}
$$

is a subalgebra of the finite SCA's.
It is of course the superalgebra of the ($\mathcal{N}$-Extended) SUSY QM.

Question: can we extend the sl(2) *D*-module reps to SCA's D-module reps?

... we already know the SQM's *D*-module reps ...

D-module reps of $\mathcal{N}$-Extended SQM

"root" supermultiplets obtained via $1D$ SUSY versus Clifford algebras connection:

$$\{Q_i, Q_j\} = \delta_{ij} H,$$

versus

$$\{\Gamma_i, \Gamma_j\} = 2\eta_{ij}.$$

The root operators Q_i can be expressed as

$$Q_i = \begin{pmatrix} 0 & \gamma_i \\ H\tilde{\gamma}_i & 0 \end{pmatrix} \begin{pmatrix} d \\ d \end{pmatrix},$$

where γ_i and $\tilde{\gamma}_i$ are antidiagonal blocks entering the Clifford algebra generators

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \tilde{\gamma}_i & 0 \end{pmatrix}, \qquad (\Gamma_i)^2 = +\mathbf{1}.$$

Root supermultiplets:

A "minimal" **root supermultiplet** (d, d) contains
d bosonic fields of scaling dimension λ and
d fermionic fields of scaling dimension $\lambda + 1/2$.

$\mathcal{N}=1$	1	$\mathcal{N}=9$	16	$\mathcal{N}=17$	256	$\mathcal{N}=25$	4096
$\mathcal{N}=2$	2	$\mathcal{N}=10$	32	$\mathcal{N}=18$	512	$\mathcal{N}=26$	8192
$\mathcal{N}=3$	4	$\mathcal{N}=11$	64	$\mathcal{N}=19$	1024	$\mathcal{N}=27$	16384
$\mathcal{N}=4$	4	$\mathcal{N}=12$	64	$\mathcal{N}=20$	1024	$\mathcal{N}=28$	16384
$\mathcal{N}=5$	8	$\mathcal{N}=13$	128	$\mathcal{N}=21$	2048	$\mathcal{N}=29$	32768
$\mathcal{N}=6$	8	$\mathcal{N}=14$	128	$\mathcal{N}=22$	2048	$\mathcal{N}=30$	32768
$\mathcal{N}=7$	8	$\mathcal{N}=15$	128	$\mathcal{N}=23$	2048	$\mathcal{N}=31$	32768
$\mathcal{N}=8$	8	$\mathcal{N}=16$	128	$\mathcal{N}=24$	2048	$\mathcal{N}=32$	32768

The mod 8 property of these reps of the $\mathcal{N}$-extended supersymmetry is in consequence of the Bott's periodicity of Clifford algebras.

Comparison: D-modules versus superspace

- For large $\mathcal{N}$ the superspace is highly reducible.
- There is no known general rule on how to impose constraints and get all minimal representations.

Number of fields (d).

$\mathcal{N}$	(a)	(b)
$\mathcal{N}=1$	1	1
$\mathcal{N}=2$	2	2
$\mathcal{N}=4$	8	4
$\mathcal{N}=8$	128	8
$\mathcal{N}=16$	2^{15}	128
$\mathcal{N}=32$	2^{31}	32768

- **(a)**: number of (bosonic) component fields in unconstrained superspace;
- **(b)**: number of (bosonic) component fields in a minimal representation.

Beyond root supermultiplets: the dressing transformations

How to obtain, e.g., the real supermultiplet: $(\phi; \psi_1, \psi_2; F) \leftrightarrow (1, 2, 1)$?

Notations

- $(d_1, d_2, \ldots, d_l)$ - *the field content* of the multiplet
- l is the *length* of the multiplet

At a given $\mathcal{N}$ the admissible $(d_1, d_2, \ldots, d_l)$ supermultiplets are recovered from the root supermultiplet (d, d) via a dressing transformation:

$$Q_i \rightarrow \widehat{Q}_i = SQ_iS^{-1}$$

For length-3 supermultiplets the $(d, d) \rightarrow (d-k, d, k)$ dressing matrix S has the form $S^{(k)} = diag(H, H, \ldots, H, 1, \ldots, 1; 1 \ldots, 1)$ (in this case H enters the diagonal k times).

For length-4 supermultiplets $(d-k, d-m, k, m)$ a dressing matrix has the form $S^{(k,m)} = diag(H, \ldots, H, 1 \ldots, 1; H \ldots, H, 1, \ldots, 1)$ with H entering the upper diagonal block k times and the lower block m times.

Admissible dressings

- For any $\mathcal{N}$, all length-3 multiplets $(d-k, d, k)$ are representations of the $\mathcal{N}$-extended supersymmetry.
- length-4 irreps are present only for $\mathcal{N} = 3, 5, 6, 7$ and for $\mathcal{N} \geq 9$, length-5 irreps appear starting from $\mathcal{N} = 10$. The results, up to $\mathcal{N} = 8$ are reported in the following table

$\mathcal{N}$	length-2	length-3	length-4
1	(1,1)		
2	(2,2)	(1,2,1)	
3	(4,4)	(3,4,1), (2,4,2), (1,4,3)	(1,3,3,1)
4	(4,4)	(3,4,1), (2,4,2), (1,4,3)	
5	(8,8)	(7,8,1), (6,8,2), (5,8,3), (4,8,4) (3,8,5), (2,8,6), (1,8,7)	(1,5,7,3), (3,7,5,1) (1,6,7,2), (2,7,6,1), (2,6,6,2), (1,7,7,1)
6	(8,8)	(7,8,1), (6,8,2), . . ., (1,8,7)	(1,6,7,2), (2,7,6,1), (2,6,6,2), (1,7,7,1)
7	(8,8)	(7,8,1), (6,8,2), . . ., (1,8,7)	(1,7,7,1)
8	(8,8)	(7,8,1), (6,8,2), . . ., (1,8,7)	

From Global susy D-modules to SCA D-modules

Input:

- A global $\mathcal{N}$-Extended supermultiplet with component fields $(d_1, d_2, d_3, \ldots)$.

Comment: the scaling dimensions of the component fields are $(\lambda, \lambda + \frac{1}{2}, \lambda + 1, \ldots)$; the overall scaling dimension of the supermultiplet is λ. In application to SCM, $\lambda \in \mathbf{R}$.

- A diagonal action of the $D, H, K \in sl(2)$ generators on the component fields, with the given scaling dimensions.

Requirement:

Closure of the (anti)commutation relations on a given finite SCA (to be determined).

Comment: the generators are obtained by repeatedly applying anti-commutators involving K and Q_i's.

E.g.: $\widetilde{Q}_i = [K, Q_i], S_{ij} = \{\widetilde{Q}_i, Q_j\}$.

For $i \neq j$, the S_{ij} generators belong to the R-symmetry algebra.

The closure on a finite SCA would require, in particular, that the commutators

$$[S_{ij}, Q_k] = \alpha^l_{ijk} Q_l,$$

close on the susy generators Q_i's (no extra odd generators).

We get a set of non-trivial relations for the consistency of SCA's D-module reps, involving the scaling dimension λ.
Comment: the S_{ij} generators, for $i > j$, are not necessarily linear independent. The number of linearly independent generators allow to identify the R-sector of the algebra.

Most relevant cases: $\mathcal{N}=4,7,8$ finite SCA's

$\mathcal{N}=4$: simple SCA's are $A(1,1)$ and the exceptional superalgebras $D(2,1;\alpha)$, for $\alpha\in\mathbb{C}\backslash\{0,-1\}$. The superalgebras $D(2,1;\alpha)$'s are isomorphic iff α are connected via an S_3 group transformation generated by the moves $\alpha\mapsto\frac{1}{\alpha}$ and $\alpha\mapsto-(1+\alpha)$. We have therefore at most 6 different α's producing, up to isomorphism, the superconformal algebra $D(2,1;\alpha)$:

$$\begin{array}{rclrclrcl}\alpha^{(1)}&=&\alpha, & \alpha^{(3)}&=&-(1+\alpha), & \alpha^{(5)}&=&-\frac{1+\alpha}{\alpha},\\ \alpha^{(2)}&=&\frac{1}{\alpha}, & \alpha^{(4)}&=&-\frac{1}{(1+\alpha)}, & \alpha^{(6)}&=&-\frac{\alpha}{(1+\alpha)}.\end{array}$$

$A(1,1)$ can be regarded as a degenerate superalgebra recovered from $D(2,1;\alpha)$ at the special values $\alpha=0,-1$ (at these special values three even generators decouple from the rest of the generators; the remaining ones close the $A(1,1)$ superalgebra).

For α real ($\alpha\in\mathbb{R}$) a fundamental domain under the action of the S_3 group can be chosen to be the closed interval

$$\alpha \quad\in\quad [0,1].$$

Some points in the interval are of special significance. We have that
i) - the extremal point $\alpha = 0$ corresponds to the $A(1,1)$ superalgebra,
ii) - the extremal point $\alpha = 1$ correspond to the $D(2,1)$ superalgebra, belonging to the $D(m|n) = osp(2m|2n)$ classical series,
iii) - the midpoint $\alpha = \frac{1}{2}$ corresponds to the $F(4)$ subalgebra $D(2,1;\frac{1}{2}) \subset F(4)$,
iv) - the point $\alpha = \frac{1}{3}$ corresponds to the $G(3)$ subalgebra $D(2,1;\frac{1}{3}) \subset G(3)$.

Over $\mathbb{C}$, there are four finite $\mathcal{N} = 8$ SCA's and one finite $\mathcal{N} = 7$ SCA.

The finite $\mathcal{N} = 8$ superconformal algebras are:
i) the $A(3,1) = sl(4|2)$ superalgebra, possessing 19 even generators and bosonic sector given by $sl(2) \oplus sl(4) \oplus u(1)$,
ii) the $D(4,1) = osp(8,2)$ superalgebra, possessing 31 even generators and bosonic sector given by $sl(2) \oplus so(8)$,
iii) the $D(2,2) = osp(4|4)$ superalgebra, possessing 16 even generators and bosonic sector given by $sl(2) \oplus so(3) \oplus sp(4)$,
iv) the $F(4)$ exceptional superalgebra, possessing 24 even generators and bosonic sector given by $sl(2) \oplus so(7)$.

The finite $\mathcal{N} = 7$ superconformal algebra is the exceptional superalgebra $G(3)$, possessing 17 even generators and bosonic sector given by $sl(2) \oplus g_2$.

Existence of critical scaling dimension λ's:

Results: D-module reps are induced, with the identifications:

$\mathcal{N}=4$: $D(2,1;\alpha)$ reps are recovered from the $(k,4,4-k)$ su[permultiplets, with a relation between α and the scaling dimension given by $\alpha=(2-k)\lambda$.

$\mathcal{N}=8$: for $k\neq 4$, all four $\mathcal{N}=8$ finite superconformal algebras are recovered, at the critical values $\lambda_k=\frac{1}{k-4}$, with the following identifications: $D(4,1)$ for $k=0,8$, $F(4)$ for $k=1,7$, $A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

$\mathcal{N}=7$: the global supermultiplet $(1,7,7,1)$ induces, at $\lambda=-\frac{1}{4}$, a D-module representation of the exceptional superalgebra $G(3)$.

Explicit non-critical $\mathcal{N}=3$ case; $B(1,1)=osp(3,2)$ from $(1,3,3,1)$:

$$
\begin{aligned}
H &= \mathbf{1}_8\cdot\partial_t,\\
D &= -\mathbf{1}_8\cdot t\partial_t-\Lambda,\\
K &= -\mathbf{1}_8\cdot t^2\partial_t-2t\Lambda,\\
Q_1 &= (-E_{38}+E_{47}-E_{52}+E_{61})\partial_t+E_{16}-E_{25}+E_{74}-E_{83},\\
Q_2 &= (E_{28}-E_{46}-E_{53}+E_{71})\partial_t+E_{17}-E_{35}-E_{64}+E_{82},\\
Q_3 &= (-E_{27}+E_{36}-E_{54}+E_{81})\partial_t+E_{18}-E_{45}+E_{63}-E_{72},\\
\widetilde{Q}_1 &= (-E_{38}+E_{47}-E_{52}+E_{61})t\partial_t+(E_{16}-E_{25}+E_{74}-E_{83})t+\\
&\quad -E_{52}(2\lambda+2)+(-E_{38}+E_{47})(2\lambda+1)+E_{61}2\lambda,\\
\widetilde{Q}_2 &= (E_{28}-E_{46}-E_{53}+E_{71})t\partial_t+(E_{17}-E_{35}-E_{64}+E_{82})t+\\
&\quad -E_{53}(2\lambda+2)+(E_{28}-E_{46})(2\lambda+1)+E_{71}2\lambda,\\
\widetilde{Q}_3 &= (-E_{27}+E_{36}-E_{54}+E_{81})t\partial_t+(E_{18}-E_{45}+E_{63}-E_{72})t+\\
&\quad -E_{54}(2\lambda+2)+(-E_{27}+E_{36})(2\lambda+1)+E_{81}2\lambda,\\
S_1 &= E_{34}-E_{43}+E_{78}-E_{87},\\
S_2 &= -E_{24}+E_{42}-E_{68}+E_{86},\\
S_3 &= E_{23}-E_{32}+E_{67}-E_{76},
\end{aligned}
$$

for $\Lambda=diag(\lambda,\lambda+1,\lambda+1,\lambda+1,\lambda+\frac{3}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2},\lambda+\frac{1}{2})$, $S_i=\epsilon_{ijk}S_{jk}$

The (anti)commutation relations of the $B(1,1)$ superalgebra reads as

$$
\begin{array}{rclrcl}
[H,K] &=& 2D, & & & \\
[D,H] &=& H, & [D,K] &=& -K, \\
[D,Q_i] &=& \frac{1}{2}Q_i, & [D,\widetilde{Q}_i] &=& -\frac{1}{2}\widetilde{Q}_i, \\
[H,\widetilde{Q}_i] &=& Q_i, & [K,Q_i] &=& \widetilde{Q}_i, \\
\{Q_i,Q_j\} &=& 2\delta_{ij}H, & \{\widetilde{Q}_i,\widetilde{Q}_j\} &=& -2\delta_{ij}K, \\
\{Q_i,\widetilde{Q}_j\} &=& -2\delta_{ij}D+\epsilon_{ijk}S_k, & [S_i,Q_j] &=& -\epsilon_{ijk}Q_k, \\
[S_i,\widetilde{Q}_j] &=& -\epsilon_{ijk}\widetilde{Q}_k, & [S_i,S_j] &=& -\epsilon_{ijk}S_k.
\end{array}
$$

D-modules and superconformale mechanics in Lagrangian framework.

The generators of the D-module representations act as graded Leibniz derivatives on functions of the component fields.

Example:

$$D(\varphi_A\varphi_B) \;=\; (D\varphi_A)\varphi_B+\varphi_A(D\varphi_B)=-\frac{d}{dt}(\varphi_A\varphi_B)-\lambda_{A\cdot B}(\varphi_A\varphi_B),$$

(the scaling dimension of $\varphi_A\cdot\varphi_B$ is $\lambda_{A\cdot B}=\lambda_A+\lambda_B$).

A manifest, global $\mathcal{N}=4$, supersymmetric action is obtained for the $(k,\mathcal{N},\mathcal{N}-k)$ supermultiplet through

$$\mathcal{L} \;=\; Q_4Q_3Q_2Q_1[F(\vec{x})],$$

where $F(\vec{x})$ is an arbitrary function (named *the prepotential*) of the propagating bosons $\vec{x}$ (this construction requires the dimensionality k of the target manifold to be $k>0$).

These Lagrangians have the correct dimensionality of a standard kinetic term, including the quadratic time-derivatives of the propagating bosons.

For $\mathcal{N}=8$, in order to produce a supersymmetric sigma-model with the correct dimensionality of the standard kinetic term, one can pick four out of the eight supercharges to be mainfestly realized. The invariance under the remaining four supercharges (Q_j, for $j=5,6,7,8$) is recovered by constraining their action on the Lagrangian to be a time-derivative:

$$Q_j\mathcal{L} \quad = \quad \frac{d}{dt}P_j$$

for some functions P_j of the component fields and their time-derivatives.

As a result the prepotential $F(\vec{x})$ is no longer unconstrained.

Explicit computations prove that the invariance under the global $\mathcal{N} = 8$ supersymmetry is achieved if F satisfies the harmonic condition

$$\Box F = 0,$$

where "$\Box$" stands for the k-dimensional Laplacian operator.

The superconformal mechanics is derived by constructing first a global supersymmetric Lagrangian and then requiring the action of K to produce a time-derivative:

$$K\mathcal{L} = \frac{d}{dt}M,$$

for some function M of the component fields and their time-derivatives.

An $\mathcal{N}=4$ SCM system: $n_\pm$ inhomogeneous $(1,4,3)$ chiral supermultiplets interacting

We illustrate here as an example the computation of the superconformal invariance for several interacting supermultiplets. Let us take n inhomogeneous $\mathcal{N}=4$ $(1,4,3)$ supermultiplets defined by the component fields (at a given $I=1,\ldots,n$) $(x^I;\psi^I,\psi_i^I;g_i^I)$, with $i=1,2,3$. Let us further assume that n_+ supermultiplets are of positive chirality ($I=1,\ldots,n_+$) and $n_-=n-n_+$ supermultiplets are of negative chirality ($I=n_++1,\ldots,n$). In terms of the global $\mathcal{N}=4$ supersymmetry the transformations of the component fields are given by

$$
\begin{array}{rclcrcl}
Q_4x^I &=& \psi^I, && Q_ix^I &=& \psi_i^I,\\
Q_4\psi^I &=& \dot{x}^I, && Q_i\psi^I &=& -g_i^I,\\
Q_4\psi_j^I &=& g_j^I, && Q_i\psi_j^I &=& \delta_{ij}\dot{x}_j^I+s_I\epsilon_{ijk}(g_k^I+c_k^I),\\
Q_4g_j^I &=& \dot{\psi}_j^I, && Q_ig_j^I &=& -\delta_{ij}\dot{\psi}^I-s_I\epsilon_{ijk}\dot{\psi}_k^I.
\end{array}
$$

The signs $s_I=\pm1$ define the chirality of the corresponding supermultiplet. The inhomogeneous parameters c_k^I are arbitrary.

Up to a total derivative, the Lagrangian $\mathcal{L} = Q_4 Q_3 Q_2 Q_1(F)$ derived from (**??**) can be expressed as

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{IJ} s_I F_{IJ} \left(-\dot{x}^I \dot{x}^J + \psi^J \dot{\psi}^I + \psi_i^J \dot{\psi}_i^I - g_i^J (g_i^I + c_i^I) \right) + \\
& \sum_{IJK} F_{IJK} \left(-s_I \psi^K \psi_i^J (g_i^I + c_i^I) + \frac{1}{2} \epsilon_{ijk} g_k^K \psi_j^J \psi_i^I) \right) + \\
& \sum_{IJKL} F_{IJKL} \left(\frac{1}{6} \epsilon_{ijk} \psi^L \psi_k^K \psi_j^J \psi_i^I \right)
\end{aligned}
$$

(here $F_I \equiv \frac{\partial}{\partial x_I} F$ and similarly for the higher order derivatives). The summation over i, j, k is understood.

This action is globally $\mathcal{N} = 4$-invariant.

By inserting the prepotential

$$F = \sum_{MN} A_{MN}\, x_M \ln x_N,$$

where A_{MN} is an arbitrary constant matrix, we obtain that the system of $n_\pm$ interacting supermultiplets is superconformally invariant under the $A(1,1)$ superalgebra (each supermultiplet has conformal weight $\lambda = -1$). The constraint induced by K gets translated into the set of constraints, for $r = 2, 3, 4$,

$$x^M F_{MI_1\ldots I_r} = \gamma^{(r)} F_{I_1\ldots I_r}, \qquad \gamma^{(r)} = 1 - r,$$

which are obviously satisfied for the above choice of F.

In the classical framework the real action (for β real)

$$\mathcal{S} = \int dt\mathcal{L} = \int dt(x^{\beta}\dot{x}^2)$$

is scale-invariant and dimensionless provided that the scaling dimension λ for the field x satisfies the condition

$$\lambda = -\frac{1}{\beta+2}$$

(the scaling dimension of t is assumed to be $[t]=-1$).
The $\mathcal{N}=8$ $(1,8,7)$ model with a unique propagating boson x, fermions ψ, ψ_j and auxiliary fields g_j ($j=1,2,\ldots,7$) is given by the action

$$\mathcal{S} = = \int dt\{(ax+b)[\dot{x}^2 - \psi\dot{\psi} - \dot{\psi}_j\psi_j + {g_j}^2] + a[\psi\psi_j g_j - \frac{1}{2}C_{ijk}g_j\psi_j\psi_k]\},$$

for some real coefficients a, b.
The connection between $\mathcal{N}=8$ supersymmetry and octonions implies that, without loss of generality, the totally antisymmetric coupling constants C_{ijk} can be identified with the octonionic structure constants. A consistent choice is $C_{123} = C_{147} = C_{165} = C_{257} = C_{354} = C_{367} = 1$.

The scale-invariance and the dimensionless of the action requires the homogeneity of the Lagrangian. Therefore, either we have $a = 0$ or $b = 0$. In the $a = 0$ (for $b \neq 0$) case we obtain a constant kinetic term. The scaling dimension λ of x coincides with the scaling dimension of the $(1, 8, 7)$ supermultiplet. It is given, for $\beta = 0$, by $\lambda = -\frac{1}{2}$. This value, however, does not coincide with the critical scaling dimension for the $\mathcal{N} = 8$ supermultiplet with $k = 1$.

In the second case ($b = 0$ and $a \neq 0$) we obtain a nontrivial Lagrangian, due to the presence of the cubic term. The scaling dimension λ is now recovered for$\beta = 1$. We obtain for this value the critical scaling dimension $\lambda = -\frac{1}{3}$ of the $\mathcal{N} = 8$ $k = 1$ supermultiplet. At this critical value the $(1, 8, 7)$ supermultiplet induces a D-module representation of the $F(4)$ superconformal algebra and the model is $F(4)$-invariant.

The S_3 α-orbit of $D(2,1;\alpha)$ and the constraints on multiparticle superconformal mechanics

The combined properties of having different α's producing isomorphic $\mathcal{N}=4$ superconformal algebras (**??**), together with the set of critical relations between α and the scaling dimension λ of the $(k,4,4-k)$ $\mathcal{N}=4$ supermultiplets (with $k=0,1,2,3,4$), given by

$$\alpha = (2-k)\lambda,$$

produce highly non-trivial constraints on the admissible $\mathcal{N}=4$ superconformal mechanics models and their scaling dimension. For the $\alpha=0,-1$ case, e.g., we have that the solutions are recovered for any real λ for the $(2,4,2)$ supermultiplet (for $\lambda\neq 0$ the superalgebra is $sl(2|2)$, while, for $k\neq 2$, they are obtained for $\lambda=0$ or $\lambda=\frac{1}{k-2}$. It is convenient to summarize some results in a table presenting the admissible scaling dimension λ associated to the $(k,4,4-k)$ supermultiplets for the above four cases, specified by α_{FD} (the value α in the fundamental domain) given by, respectively, $\alpha_{FD}=0,1,\frac{1}{2},\frac{1}{3}$.

$\alpha_{FD} = 0$:

k	λ
0	$0, -\frac{1}{2}$
1	$0, -1$
2	$\mathbb{R}$
3	$0, 1$
4	$0, \frac{1}{2}$

$\alpha_{FD} = 1$:

k	λ
0	$-1, -\frac{1}{4}, \frac{1}{2}$
1	$-2, -\frac{1}{2}, 1$
3	$-1, \frac{1}{2}, 2$
4	$-\frac{1}{2}, \frac{1}{4}, 1$

	k	λ
$\alpha_{FD} = \frac{1}{2}$:	0	$-\frac{3}{2}, -\frac{3}{4}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{4}, 1$
	1	$-3, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}, 2$
	3	$-2, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3$
	4	$-1, -\frac{1}{4}, \frac{1}{6}, \frac{1}{3}, \frac{3}{4}, \frac{3}{2}$

	k	λ
$\alpha_{FD} = \frac{1}{3}$:	0	$-2, -\frac{2}{3}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{6}, \frac{3}{2}$
	1	$-4, -\frac{4}{3}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{3}, 3$
	3	$-3, -\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, 4$
	4	$-\frac{3}{2}, -\frac{1}{6}, \frac{1}{8}, \frac{3}{8}, \frac{2}{3}, 2$

The $\mathcal{N}=4$ superconformal invariance for several (at least two, let's say $(k,4,4-k)$ and $(k',4,4-k')$) interacting supermultiplets requires that they should carry a D-module representation for the same $D(2,1;\alpha)$ superalgebra. Given two supermultiplets with $k'\neq k$, this requirement produces
a) a constraint on the admissible values for α and
b) a consequent constraint on the mutual scaling dimensions of the two supermultiplets (both these constraints will be called the "compatibility condition").
The table above shows that, for $\alpha_{FD}=0,1,\frac{1}{2}$, two supermultiplets with $k'\neq k$ and the same scaling dimension λ can be found for

$$\begin{array}{llllll}
\alpha_{FD}=0: & \lambda=0 & (k,k'\neq 2), & \lambda=0,\frac{1}{k-2} & (k'=2),\\
\alpha_{FD}=1: & \lambda=-1,\frac{1}{2} & (k=0,k'=3), & \lambda=1,-\frac{1}{2} & (k=1,k'=4),\\
\alpha_{FD}=\frac{1}{2}: & \lambda=-\frac{3}{2},-\frac{1}{3} & (k=0,k'=1), & \lambda=\frac{3}{2},\frac{1}{3} & (k=3,k'=4).
\end{array}$$

Some conclusions can be drawn from this result. For instance, the decomposition of $\mathcal{N}=8$ D-module representations into $\mathcal{N}=4$ supermultiplets can (partly) explain the $\mathcal{N}=8$ critical scaling dimensions. The $(7,8,1)$ supermultiplet gets decomposed into $(4,4,0)\oplus(3,4,1)$. Its critical scaling dimension is therefore constrained to be either $\lambda=\frac{3}{2}$ or $\lambda=\frac{1}{3}$ (its actual value). The $(4,4,0)\oplus(2,4,2)$ decomposition of the $(6,8,2)$ supermultiplet implies that its critical scaling dimension can only be found at $\lambda=0$ or $\lambda=\frac{1}{2}$ (its actual value). The $(5,8,3)$ supermultiplet admits the decompositions $(4,4,0)\oplus(1,4,3)$ and $(3,4,1)\oplus(2,4,2)$. Their combination uniquely implies a possible critical scaling dimension at $\lambda=1$ (its actual value).

The $\alpha_{FD}=\frac{1}{3}$ case does not admit a common scaling dimension λ if $k\neq k'$. This value of α corresponds to the decomposition of the $\mathcal{N}=7$ $(1,7,7,1)$ supermultiplet into the $\mathcal{N}=4$ $(1,4,3,0)\oplus(0,3,4,1)$ supermultiplets. Let us denote with λ_1,λ_3 the respective scaling dimensions. Clearly $\lambda_3=\lambda_1+\frac{1}{2}$. The unique pair of values differing by $\frac{1}{2}$ are recovered for $\lambda_1=-\frac{1}{4}$ and $\lambda_3=\frac{1}{4}$. Indeed, $\lambda=-\frac{1}{4}$ is the critical scaling dimension of the $(1,7,7,1)$ D-module representation of $G(3)$.

Most general case of finding the compatibility conditions on α and the common scaling dimension λ for two $\mathcal{N}=4$ D-module representations with $k \neq k'$: it is sufficient to discuss the $k, k' \neq 2$ restriction, since the remaining cases are immediately recovered from the λ solutions at $\alpha = 0, -1$. Without loss of generality we can set $\alpha \equiv \alpha^{(1)} = (2-k)\lambda$ for the k supermultiplet. The α' value obtained as $\alpha' = (2-k')\lambda$ from the k' supermultiplet must coincide with one of the $\alpha^{(i)}$ in the S_3-orbit of α. Let us introduce the ratios $N^{(i)} = \frac{\alpha^{(i)}}{\alpha^{(1)}}$ and $w_{kk'} = \frac{2-k'}{2-k} = \frac{1}{w_{k'k}}$.
The values obtained by $w_{kk'}$ in varying k, k' with the given constraints are $-1, \pm\frac{1}{2}, \pm 2$.
For a given pair $[k, k']$ the admissible values α satisfying the compatibility condition are recovered by $\overline{\alpha}$ and its S_3-group orbit with $\overline{\alpha}$ a solution of one of the equations

$$N^{(i)} \quad = \quad w$$

(since no confusion will arise, for simplicity, we set $w \equiv w_{kk'}$).

The compatibility conditions are recovered for three inequivalent values of w, given by $w = -1, -2, 2$, due to the fact that the transformation $w \leftrightarrow \frac{1}{w}$ reflects the $k \leftrightarrow k'$ exchange.
Two of the equations are linear in α, while the three remaining ones are quadratic (producing, in some cases, complex solutions).
The $\overline{\alpha}$ solutions can be divided into three classes: real and rational, real and irrational, complex.
The complex solutions (associated to scaling dimension λ's which do not satisfy the reality condition) are found to be

$$\begin{aligned} w = -1 &\quad : \quad \overline{\alpha} = \pm i, \\ w = -2 &\quad : \quad \overline{\alpha} = \pm \frac{i}{\sqrt{2}}, \\ w = 2 &\quad : \quad \overline{\alpha} = \frac{1}{2}(-1 \pm i), -1 \pm i\sqrt{7}. \end{aligned}$$

In the rational case, the unique solutions are encountered for $w = -2$ (the $\overline{\alpha}$ S_3-orbit is specified by $\alpha_{FD} = 1$) and $w = 2$ (with orbit specified by $\alpha_{FD} = \frac{1}{2}$). We therefore recover the solutions already encountered in (**??**) and their corresponding scaling dimension λ's. No further rational solution is found.

For the irrational case we have

$[k,k']$	w	α_{FD}	λ
$[1,3]$	-1	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	$-\frac{1}{2}\pm\frac{\sqrt{5}}{2},\ \frac{1}{2}\pm\frac{\sqrt{5}}{2}$
$[0,4]$	-1	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	$-1\pm\sqrt{5},\ 1\pm\sqrt{5}$
$[1,4]$	-2	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
$[3,0]$	-2	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
$[1,0]$	2	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$
$[3,4]$	2	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
$[1,0]$	2	$\sqrt{2}-1$	$-\sqrt{2}$
$[3,4]$	2	$\sqrt{2}-1$	$\sqrt{2}$

Unlike the $\alpha_{FD}=1$ with $\lambda=1$ (for $[k,k']=[1,4]$) and $\lambda=-1$ (for $[k,k']=[0,3]$) and $\alpha_{FD}=\frac{1}{2}$ with $\lambda=\frac{1}{3}$ (for $[k,k']=[3,4]$) and $\lambda=-\frac{1}{3}$ (for $[k,k']=[0,1]$), the irrational cases and the remaining rational cases do not allow the extension of the $\mathcal{N}=4$ superconformal invariance to a broader $\mathcal{N}=8$ superconformal invariance.

A particularly interesting case involves the irrational solution of the $k = 1$, $k' = 3$ supermultiplets.

The value α_{FD} coincides with the golden mean conjugate $\Phi = -\frac{1}{2}(1-\sqrt{5})$ (the golden mean $\varphi = \frac{1}{2}(1+\sqrt{5})$ belongs to its S_3-orbit). This case admits four compatible solutions for the scaling dimension λ (given by $\pm\phi$ and $\pm\Phi$).

There exists an $\mathcal{N} = 4$ superconformal mechanics of interacting supermultiplets whose common scaling dimension is given by the golden ratio.

Classical versus quantum D-module representations

The extension to quantum mechanics can be achieved in at least two different ways. The Lagrangian mechanics can be reformulated in the Hamiltonian framework, so that standard methods of quantization can be applied, at least in principle, to the classical Hamiltonian dynamics. A more direct approach (the one we discuss here) consists in realizing the generators of the D-module representations as Hermitian operators. It is convenient to express the Hermitian $sl(2)$ generators D, H, K acting on a given component field as

$$H = p_t, \quad D = -(tp_t + i\lambda), \quad K = -(t^2 p_t + 2i\lambda t)$$

(the constraint on the scaling dimension λ will be determined in the following), while the "quantum" D-module representation for the Q_i's is obtained from the classical one by replacing the $\pm\frac{d}{dt}$ entries ($\pm\frac{d}{dt} \rightarrow \pm p_t$), while leaving unchanged the c-number entries (± 1). We will see that this is the correct prescription to obtain Hermitian global supercharges.

Hermiticity properties of the dilatation operator D and of the global supercharges Q_i. We require in particular that, acting on given supermultiplets $|m_j>$, the equalities

$$\begin{aligned} \int dt < m_1|\eta|Dm_2 > &= \int dt < Dm_1|\eta|m_2 >, \\ \int dt < m_1|\eta|Q_i m_2 > &= \int dt < Q_i m_1|\eta|m_2 >, \end{aligned}$$

(with η a given metric to be specified) have to be satisfied.
For simplicity, take $|m_1>=|m_2>\equiv|m>$ case and $|m>$ a $(k, \mathcal{N}, \mathcal{N}-k)$ supermultiplet for $\mathcal{N}=4,8$. The component fields in the $|m>$ supermultiplets are x_l $(l=1,\ldots,k)$, g_m $(m=1,\ldots,\mathcal{N}-k)$ and the fermionic (anticommuting) fields ψ_n $(n=1,\ldots,\mathcal{N})$. A constant metric η can be chosen to be $\eta = diag(1,\ldots,{p_t}^2,\ldots,p_t,\ldots)$ with the 1 entry repeated k times, the ${p_t}^2$ entry repeated $\mathcal{N}-k$ times and the p_t entry repeated $\mathcal{N}$ times. The global supercharges Qi's, recovered from the classical ones with the prescription introduced before are Hermitian.

For what concerns the dilatation operator D, the requirement of being Hermitian implies constraints on the scaling dimensions λ_x, λ_g and λ_ψ of the component fields x_l, g_m and ψ_n, respectively. We obtain

$$\lambda_x + {\lambda_x}^* = 1, \qquad \lambda_g + {\lambda_g}^* = -1, \qquad \lambda_\psi + {\lambda_\psi}^* = 0.$$

The hermiticity condition for the scaling dimension λ_x (associated with the metric $\eta = 1$) implies that λ_x belongs to the critical strip

$$\lambda_x = \frac{1}{2} + i\gamma, \qquad (\gamma \in \mathbb{R}).$$

This is the critical strip where the non-trivial zeroes of the Riemann's zeta function are encountered. This fact is at the core of a well-known strategy which has been elaborated for proving the Riemann's conjecture by linking it with the hermiticity property of the dilatation operator.

The hermiticity condition implies λ_g, λ_ψ belonging to the strips $\lambda_g = -\frac{1}{2} + i\gamma'$ and $\lambda_\psi = i\gamma''$ (with $\gamma', \gamma'' \in \mathbb{R}$), respectively. By setting the scaling dimensions of the component fields to be real, it turns out that they differ by $\frac{1}{2}$ ($\lambda_\psi = \lambda_g + \frac{1}{2}, \lambda_x = \lambda_\psi + \frac{1}{2}$) as it should be, also in accordance with the classical analysis.

The hermiticity conditions depend on the choice of the metric η, which is not necessarily constant. Take a single $|x>$ with scaling dimension λ. In the classical framework the real action (for β real) $\mathcal{S} = \int dt(x^\beta \dot{x}^2)$ is scale-invariant and dimensionless for $\lambda = -\frac{1}{\beta+2}$.
Its quantum counterpart is the hermiticity condition
$\int dt < x|\eta|Dx >= \int dt < Dx|\eta|x >$ for a non-constant metric η of the form

$$\eta \; = \; A\eta_1 + B\eta_2, \qquad \eta_1 = p_t x^\beta p_t, \qquad \eta_2 = {p_t}^2 x^\beta + x^\beta {p_t}^2,$$

with A, B some real constants.
The fulfilling the hermiticity condition implies the vanishing of the coefficients a, b multiplying two types of terms (the only ones surviving after integration by parts), given by

$$\left(\int dt < x|\eta|Dx >= \int dt < Dx|\eta|x >\right) \quad \Longrightarrow \quad \left(a\int dt(x^\beta \dot{x}^2) + b\int dt(tx^{\beta-1}\dot{x}^3) = 0\right).$$

The vanishing of b fixes the relative coefficient between A and B:

$$A = -\beta N, \qquad\qquad B = N.$$

The vanishing of a requires λ to satisfy the condition $\lambda + \lambda^* = -\frac{2}{\beta+2}$.

Thanks for the attention.