Actual Divergence of perturbative QCD series at Low Energy, II

[The AS Summation, APT & MPT models for QCD]

Dmitry V. SHIRKOV

Bogoliubov Lab, JINR, Dubna

Asympt.Series (AS) born by Essential Singularity $e^{-1/g}$

The singularity $e^{-1/g}$ is usual in Theory of Big Systems (representable via Functional or Path Integral) :

- **Turbulence**
- **Classic and Quantum Statistics**
- **Quantum Fields**

Reason : small parameter $g << 1$ at nonlinear structure

- **Energy Gap in SuperFluidity and SuperConductivity**
- **Tunneling in QM**
- **Quantum Fields (Dyson singularity), ...**

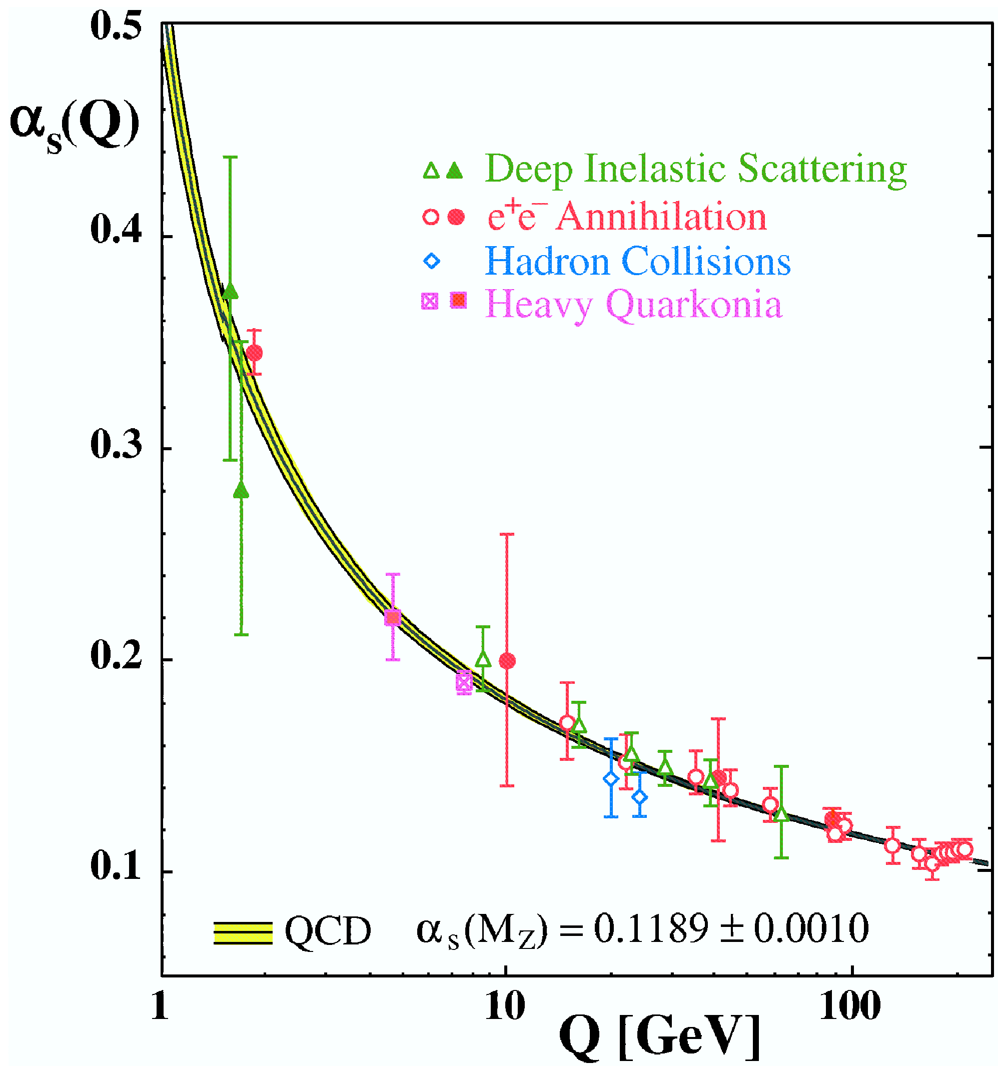
Generally, a certain AsymptSeries can correspond to a set of various functions.

. Their "summation" is an Art.

Dangerous domain for the pQCD

In QFT, all observables being renorm-invariant are expressible via RG-invariant coupling function; in perturb. QCD case – in the form of Taylor series in powers of strong "running" coupling $\alpha_s(Q)$. Due to non-abelian anti-screening, it decreases with the momentum-transfer Q increase (asymptotic freedom). Accordingly, $\alpha_s(Q)$ grows up to 0.3-0.4 values at $Q \sim 1 - 2\,GeV$ =

= Dangerous domain !



S.Bethke 2006 review

Perturb QCD contribution to Bjorken SR blows up

$$\Gamma_1(Q^2) = \frac{g_A}{6}\left[1 - \Delta^{PT}(Q^2)\right] + \Gamma_{HT}\,; \;, \qquad (1)$$

is known now up to the 4-loop term

$$\Delta^{PT} = \frac{\alpha_s(Q)}{\pi} + 0.363\alpha_s^2(Q) + 0.652\alpha_s^3(Q) + 1.804\,\alpha_s^4(Q) \qquad (2)$$

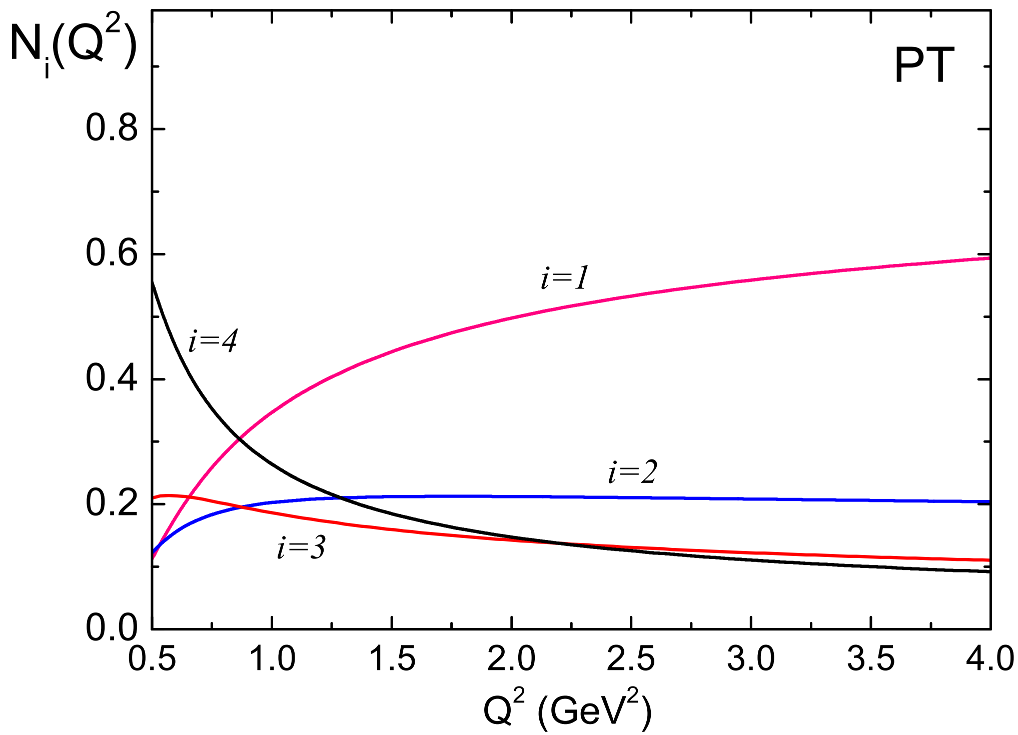
with the coefficient ratios [1 : 1 : 2 : 6] close to the factorial ones !

There are precise JLab data at very low Q values.

However, PT series "blows up"

at $Q \lesssim 1.5 - 2\,\mathrm{GeV}$;

$\alpha_s(1.5) \sim 0.4; \quad \alpha_s(2) \sim 0.3$

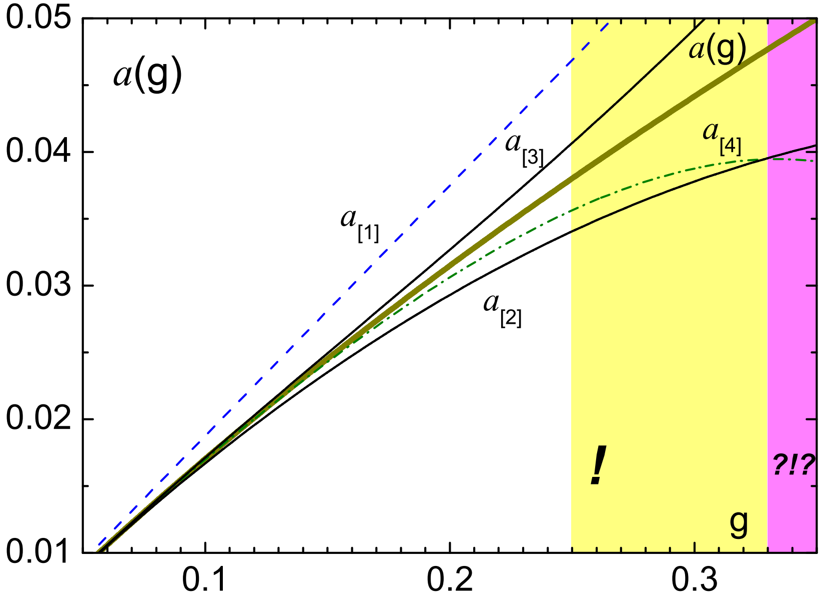


Relative weight of 1-, 2-, 3-, 4-loop terms.

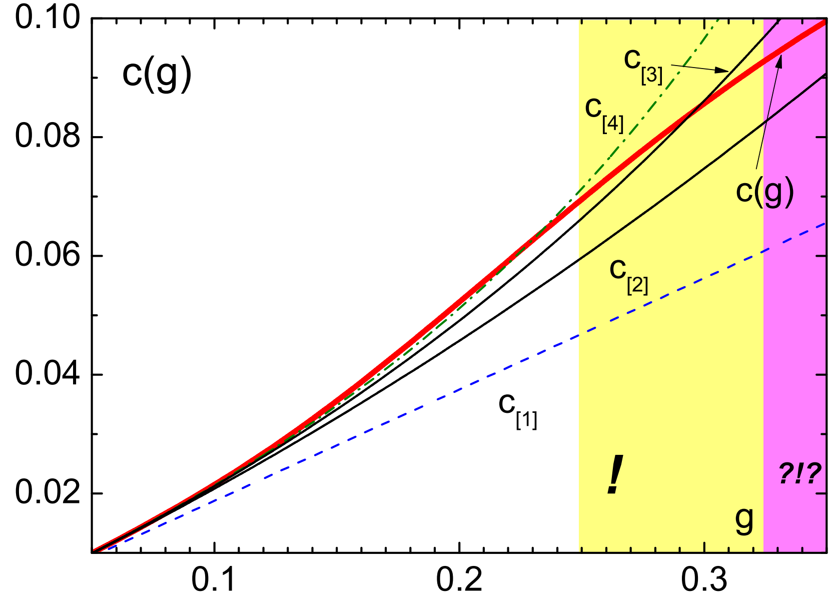
The Lessons of two Illustrations

$$a(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2-(g/4)\,x^4}\, dx - 1\,; \;\; c(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2(1-\frac{\sqrt{g}}{4}\,x)^2}\, dx - 1\,; \qquad g > 0$$

$$c/a(g) \;\to\; g \pm g^2 + 2\,g^3 \pm 6\,g^4$$



The $a_{[k]}$ approximants for function $A(g)$.



The $c_{[k]}$ approximants for $C(g)$.

It is staggering that both the examples – alternating & non-alternating – close follow not only the transparent "critical order rule" $\mathrm{K} \sim 1/\mathrm{g}$ but more subtle the "Poincaré error estimate" $\Delta\mathrm{F}(\alpha_s) \sim \mathrm{f_K}$ as well.

Theoretical prediction of higher coefficients

A good old example: **The $g\,\phi^4$ beta-function was known up to the N^3LO term**

$$\beta_{\overline{\rm MS}} = \frac{3}{2}\,g^2 - \frac{17}{6}g^3 + 16.27\,g^4 - 135.8\,g^5\,.$$

The Kazakov-Sh.-80 "summed" [by Conform-Borel method] expression

$$\beta^{CB}_{\overline{\rm MS}}(g) = \int_0^\infty \frac{dx}{g} e^{-x/g} \left(\frac{d}{dx}\right)^5 B(x) \quad \textbf{with} \qquad (3)$$

$$B(x) = a\,x^2(1 - b_2 w + \cdots - b_4 w^3); \quad w(x) - \textbf{conform variable}\,;$$

contains N^4LO term $\beta^{CB}_6 = 1409.6\,.$

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contains N^4LO term $\beta_6^{CB} = 1409.6\,.$ Soon, it was calculated via Feynman diagrams. Comparing of $\beta_6 = 1420.6$ gives the accuracy of the (4) prediction – within 1 % ! !

Higher PT contributions to observables

Relative contributions (in %) of

1– , 2–, 3– and 4–loop terms

Process		**Scale/Gev**	***PT (in %)***			
the loop number =			**1**	**2**	**3**	**4**
Bjorken SR	**t**	**1**	**35**	**20**	**19**	**26**
Bjorken SR	**t**	**1.78**	**56**	**21**	**13**	**11**
GLS SumRule	**t**	**1.78**	**58**	**21**	**12**	**11**
Incl. τ-decay	**s**	**1.78**	**51**	**27**	**14**	**7**

Relative contributions of 1- ... 4–loop

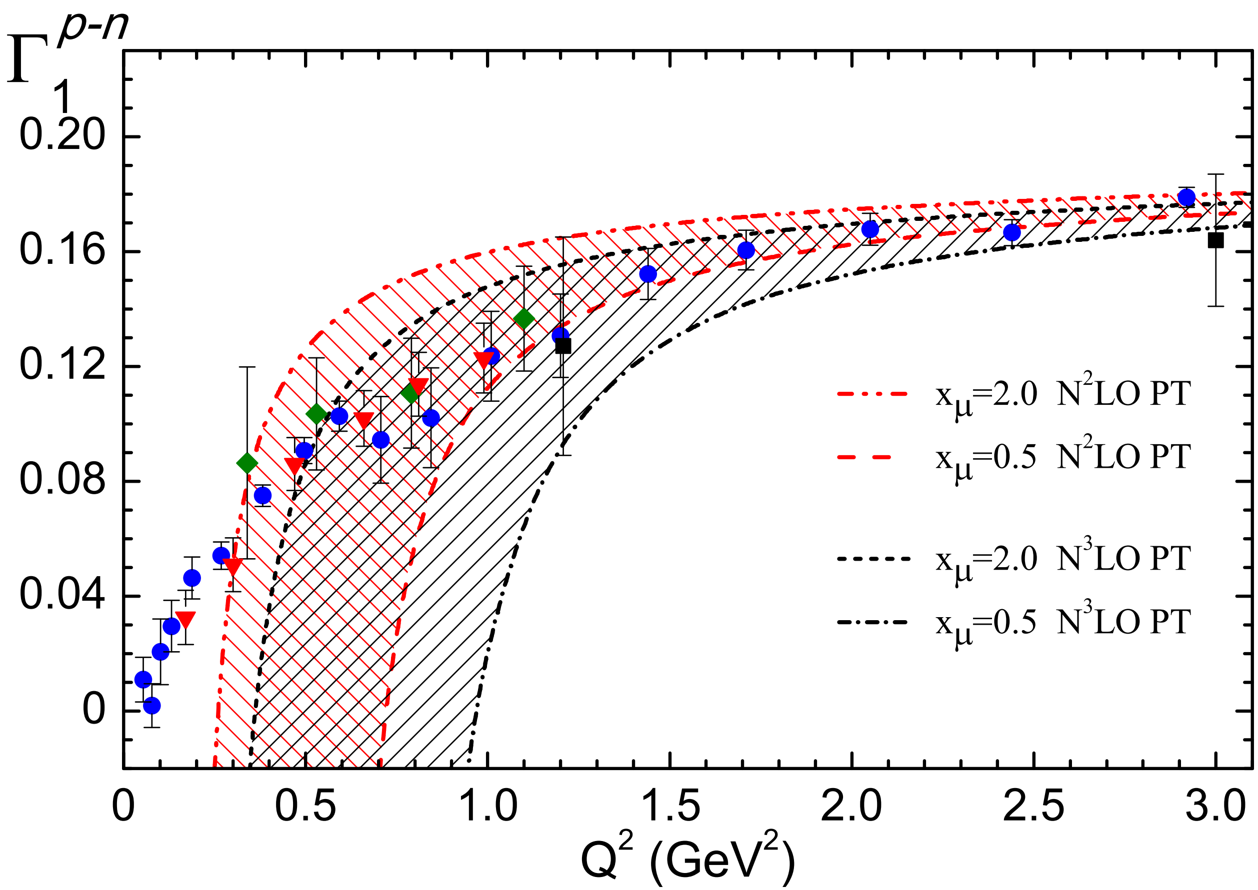
terms in $e^+e^- \to$ hadrons

Function	**Scale/Gev**	***PT terms (in %)***				**Comment**
the loop number =		**1**	**2**	**3**	**4**	
r(s)	**1**	**65**	**19**	**55**	**– 39**	**?!?**
r(s)	**1.78**	**73**	**13**	**24**	**-10**	**? !**
d(Q)	**1**	**56**	**17**	**11**	**16**	**in agenda**
d(Q)	**1.78**	**75**	**14**	**6**	**5**	**in agenda**

In the $r(s)$ **higher coefficients –**

— terrible effect of the π^2 **terms !**

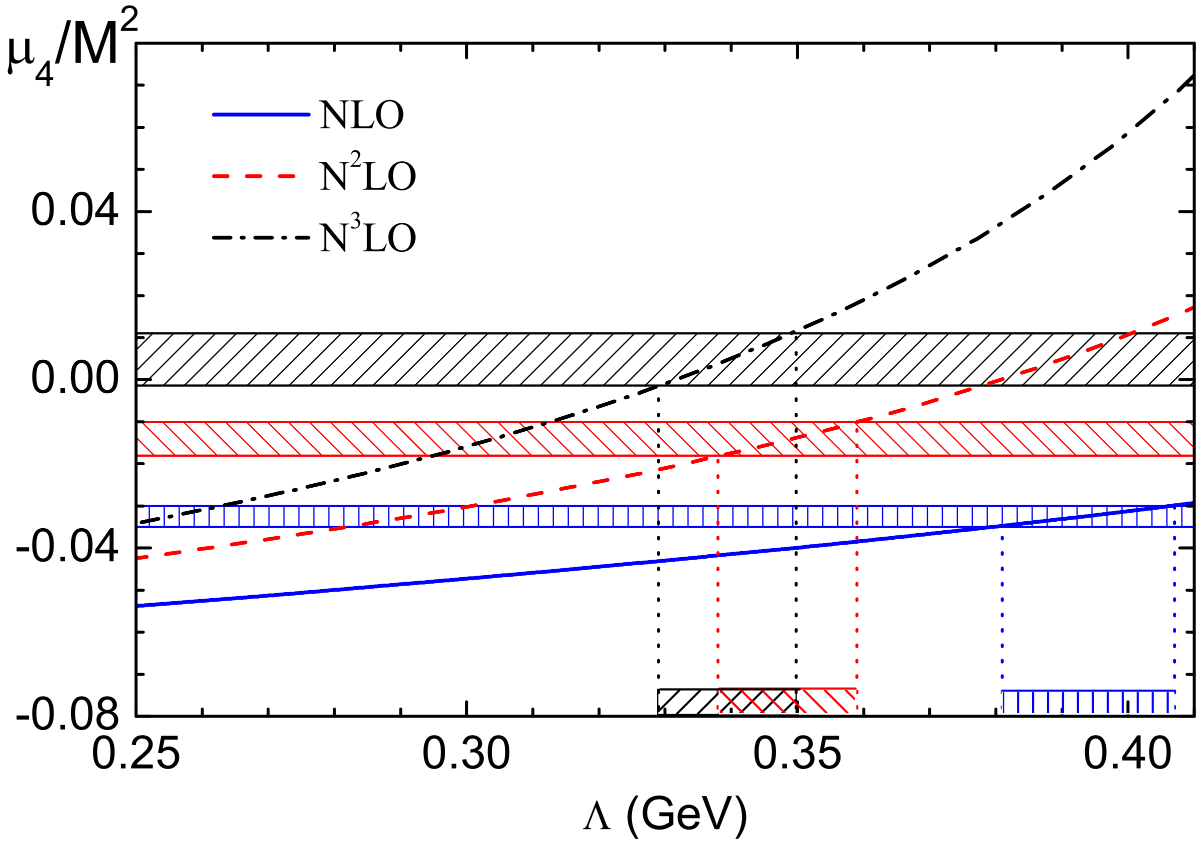
The 3- and 4-loop pQCD for Bjorken SumRule



Description of JLab data for the 1st moment Γ_1^{p-n}

4-loop fit is slightly worse than the 3-loop one

Extracting Λ_{QCD} from Bjorken SR



Extracting Λ_{QCD} from 3- and 4-loop fits for JLab data

Again no profit from the 4-loop fit !

Few words on the APT

- Non-power set of PT-expansion functions $\mathcal{A}_k(Q)$ instead of the $\alpha_s(Q)$ powers ;
- All the functions reflect RG-invariance and causality via Qr-analyticity;
- Euclidean $\mathcal{A}_k$ expansion functions are different from the Minkowskian $\mathfrak{A}_k$ ones ; all of them :
  - are related via differential recurrent relations
  - the higher functions $k \geq 2$ vanish at the IR limit ;
  - in the region above 1-2 GeV quickly tend to the α_s powers ;
- As all the expansion functions incorporate e^{-1/α_s} structures, the PT convergence improves drastically ;

Numerous applications to data analysis demonstrate the APT effectiveness in the 1 GeV region.
However, below 500 MeV the APT meets some troubles.

On the [$Q^2\, exp 1/\alpha_s$] *structure*

RG-invariance reducing the No of independent arguments,
– in the massless UV case

$$f(\ln Q^2, \alpha_s) \to F_{RGinv}\left(\frac{1}{\alpha_s} + \beta_0 \ln\left(\frac{Q^2}{\mu^2}\right)\right) = \Psi\left(\frac{Q^2}{\mu^2}\, e^{1/\beta_0\alpha_s} = \frac{Q^2}{\Lambda^2}\right);$$

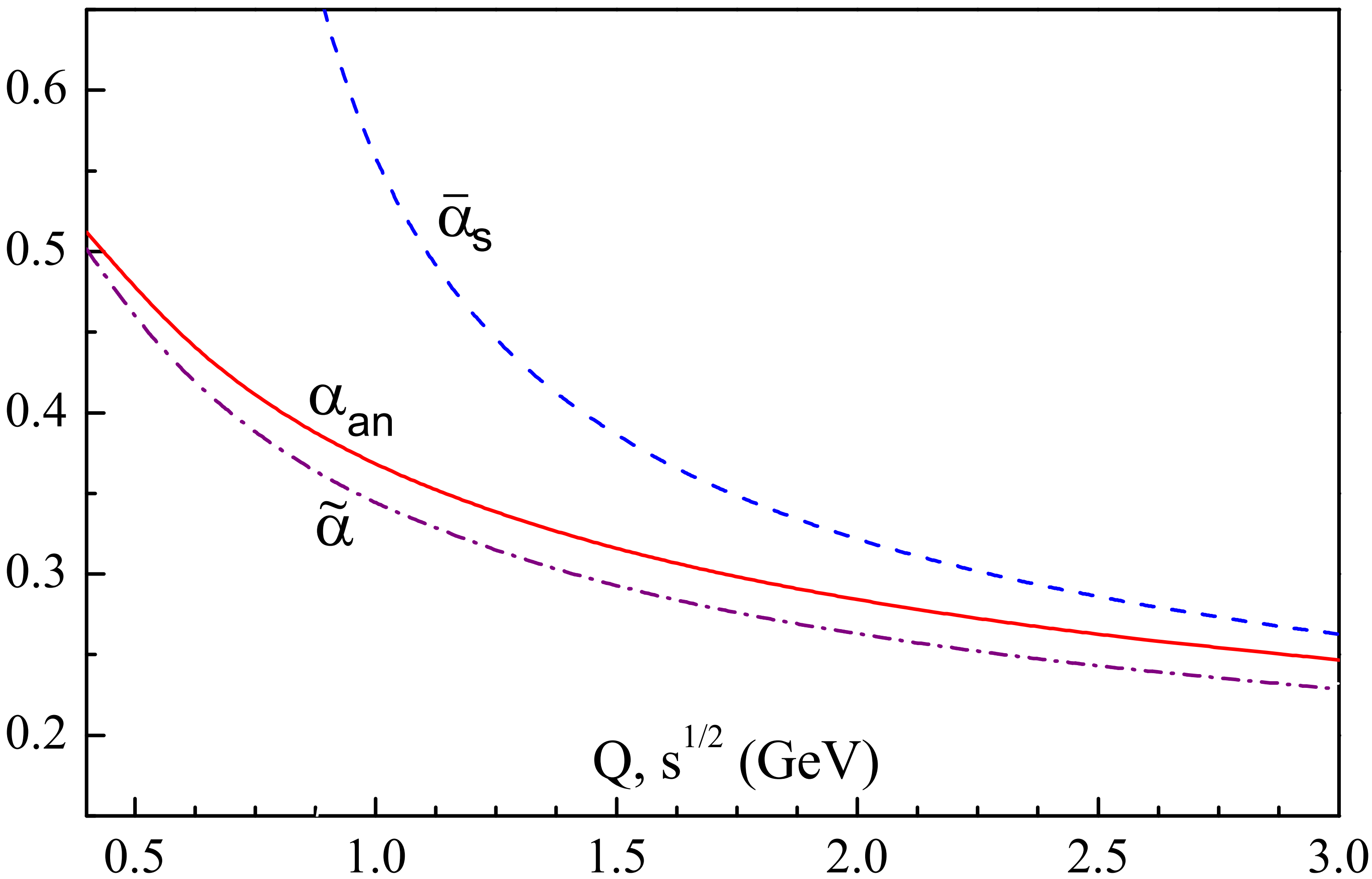
together with Q^2 analyticity yields one more statement on inevitable not-perturbative nature $\sim \mathbf{e}^{-1/\alpha_s}$ of all algebraic -in Q^2- structures, like HT terms (and singularity-killing structures in APT). E.g., at the

1-loop case $$\alpha_s(\mathbf{Q^2}) = \frac{\alpha_s}{1 + \alpha_s\beta_0\, \ln(\mathbf{Q^2}/\mu^2)} = \frac{1}{\beta_0\, \ln(\mathbf{Q^2}/\Lambda^2)} \to \mathcal{A}_1(\mathbf{Q^2});$$

$$\mathcal{A}_1(\mathbf{Q^2}) = \frac{1}{\beta_0\, \ln(\mathbf{Q^2}/\Lambda^2)} + \frac{\Lambda^2}{\beta_0\,(\Lambda^2 - \mathbf{Q^2})} = \alpha_s(\mathbf{Q^2}) + \frac{\mu^2}{\beta_0\,\left(\mu^2 - \mathbf{Q^2}\,\mathbf{e}^{1/\beta_0\alpha_s}\right)}.$$

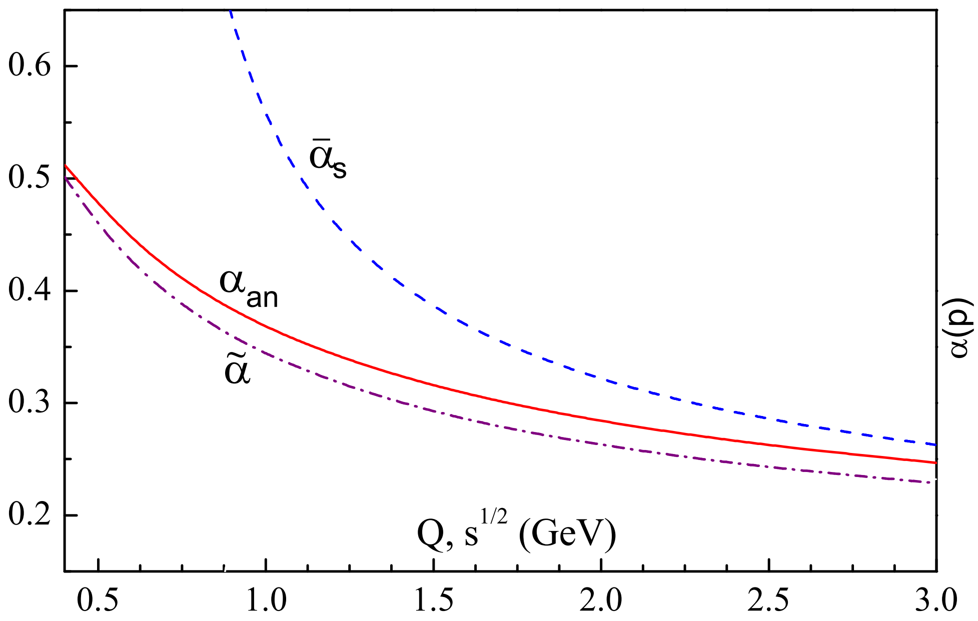
The UV log is responsible for singularity at $Q^2 = 0$.

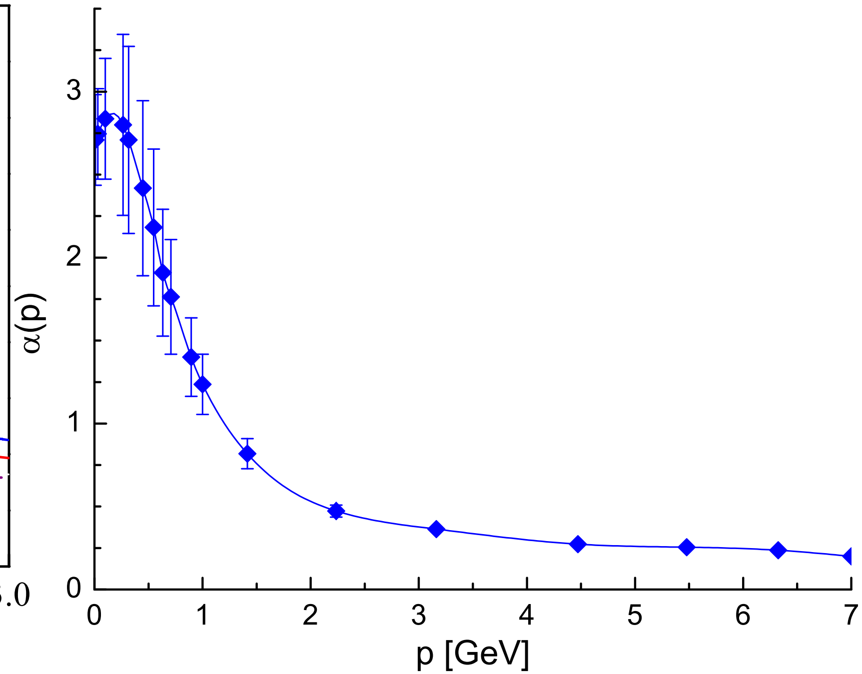
Comparing APT couplings with singular $\alpha_s(Q^2)$



Red curve – $\alpha_{an} = \alpha_{APT}(Q)$, [black dash-dotted – $\tilde{\alpha}_{APT}(\sqrt{s})$]

The APT smooth coupling vs. lattice $\alpha_s(p)$ *, below 1* GeV



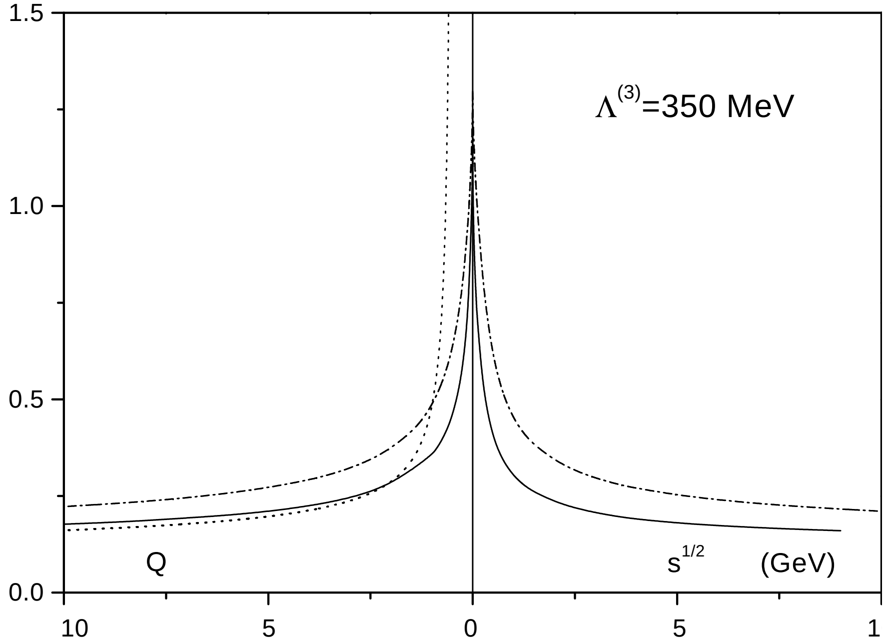


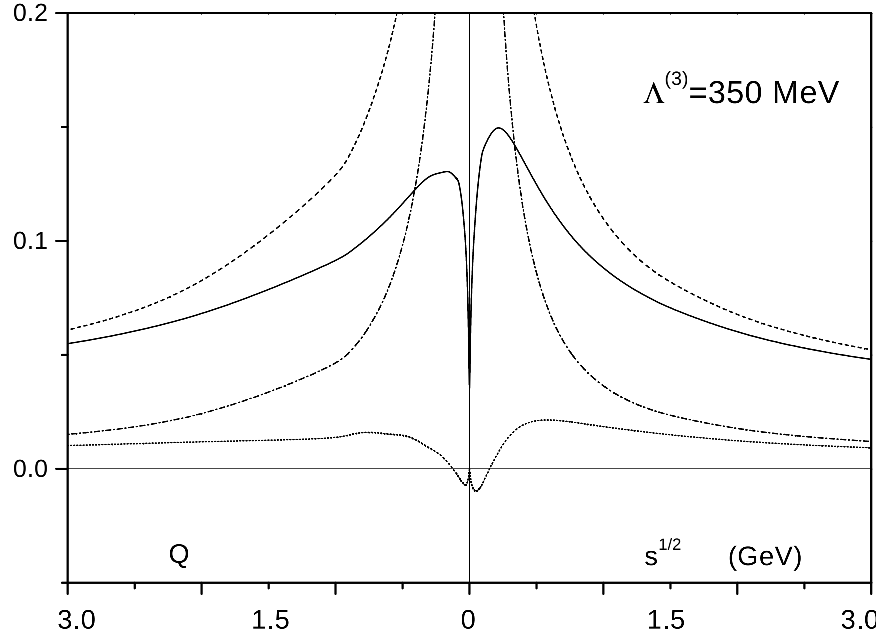
[Red line = $\alpha_{APT}(Q)$ **, black dash-dotted -** $\tilde{\alpha}_{APT}(\sqrt{s})$**]**

Lattice α_s based on ghost-gluon vertex **by [Alkofer et al., JHEP 0201,046 (2002)]**

The APT coupling has no problem with Landau singularity being finite down to IR. However, at $Q \lesssim 1\,\mathrm{GeV}$ it is smaller than lattice-simulated α_s ; besides it has infinite derivative at IR limit

Q- and s-dependence of APT functions. Distorted Mirror





Loop dependence of $\alpha_{APT}(Q)$ and $\tilde{\alpha}_{APT}(s)$

[2- and 3-loops very close each other]

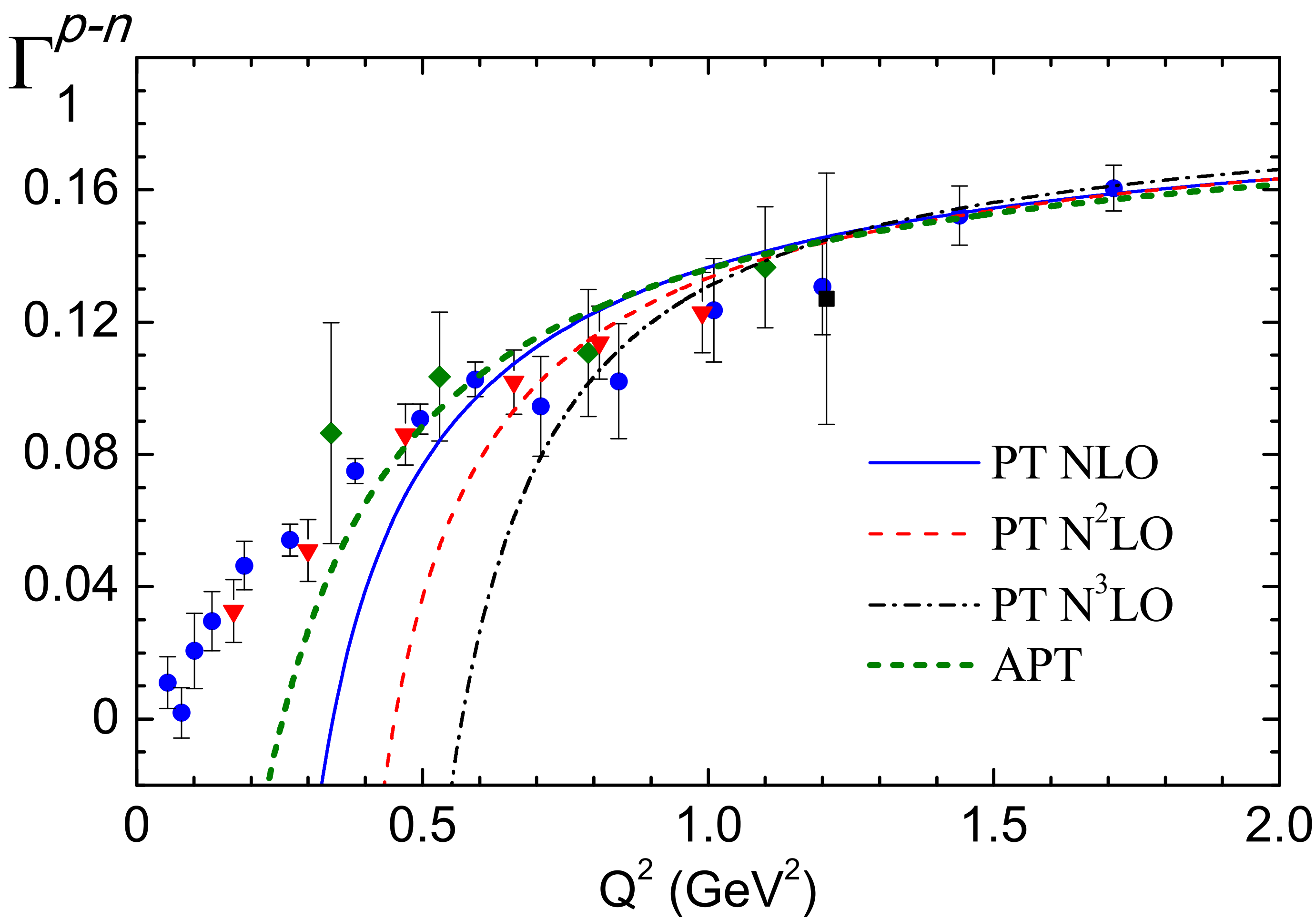
Higher APT expansion functions

[vanish at the IR limit]

An unpleasant feature one still has in APT -

the infinite derivatives at $Q^2 = 0$.

The JLab-data Description by PT and by APT+HT



Anti-progress as $2 \to 3 \to$ 4-loop PT below $Q < 1\,\mathrm{GeV}$

vs. stable APT+HT fit down to $Q^2 \sim 0.4\,\mathrm{GeV}^2$

Higher Twists for PT and APT+HT

Table 1: **HT extraction from JLab data on BSR in PT – uncertain ?**

PT	Q^2_{min},	μ_4/M^2	μ_6/M^4	μ_8/M^6
NLO	0.5	$-0.028(5)$	–	–
N^{2}LO	0.66	$-0.014(7)$	–	–
N^{3}LO	0.66	$0.005(9)$	–	–

Table 2: **HT extraction from JLab data in APT – stable !.**

APT	Q^2_{min}, **GeV^2**	μ_4/M^2	μ_6/M^4	μ_8/M^6
NLO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$
N^{2}LO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$
N^{3}LO	0.078	$-0.061(4)$	$0.009(1)$	$-0.0004(1)$

The proposed *“massive analytic pQCD”* **= MPT is constructed on the two grounds.**

*** One is the pQCD itself with one parameter added, the effective “glueball mass”,** $m_{gl} \lesssim 1\,\mathrm{GeV}$ **serving as an IR regulator.**

**** The second stems out of the ghost-free Analytic Perturbation Theory (= APT) comprising**

Non-power perturbative expansion

that makes it compatible with linear integral transformations.