Actual Divergence of perturbative QCD series at Low Energy, I

[Divergent Series, Summation,]

Dmitry V. SHIRKOV

Bogoliubov Lab, JINR, Dubna

is a general phenomenon in current theory.

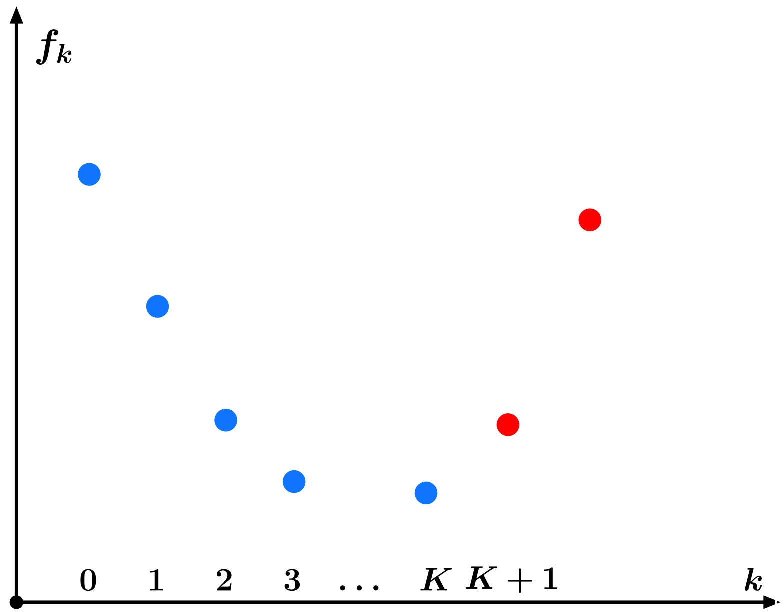
An illustration – Formal Divergent series $F(g) \sim \sum_n n!\,(g)^n$.

Its Finite Sum

$$F_k(g) = \sum_n^k f_n; \quad f_n = n!\,g^n$$

as Poincaré proved can serve for numerical estimate with the error

$$\boxed{\Delta F(\alpha_s) \sim f_K}$$



Hence, there exists Critical number of terms $K \sim 1/g$ for Optimal error = lower limit of accuracy, $\mathbf{f_K}$.

4-loop Suspicion for the Bjorken and Gross-LluvSmith Sum Rules

As it was recently calculated [Chetyrkin et al,. 2010], the 4-loop pQCD expansions for the BjSR and GLS-SR

$$\Delta^{PT}_{Bj} = \frac{\alpha_s(Q)}{\pi} + 0.363\alpha_s^2(Q) + 0.652\alpha_s^3(Q) + 1.804\,\alpha_s^4(Q) \quad (1)$$

$$\Delta^{PT}_{GLS} = \frac{\alpha_s(Q)}{\pi} + 0.363\alpha_s^2(Q) + 0.612\alpha_s^3(Q) + 1.647\,\alpha_s^4(Q) \quad (2)$$

resemble factorial series just discussed as the coefficient ratios are close numerically to [1 : 1 : 2 : 6], the factorial ones. Meanwhile, the common pQCD running coupling $\alpha_s(Q)$ takes the values $\alpha_s(1.77]\,\mathrm{GeV}) = 0.34$ and $\alpha_s(1\,\mathrm{GeV}) \sim 0.5\,.$ Now, according to the Poincaré rule, the optimal numbers of terms are

$$K(1.78\,\mathrm{GeV}) = 3 \quad \textbf{and} \quad K(1\,\mathrm{GeV}) = 2\,,$$

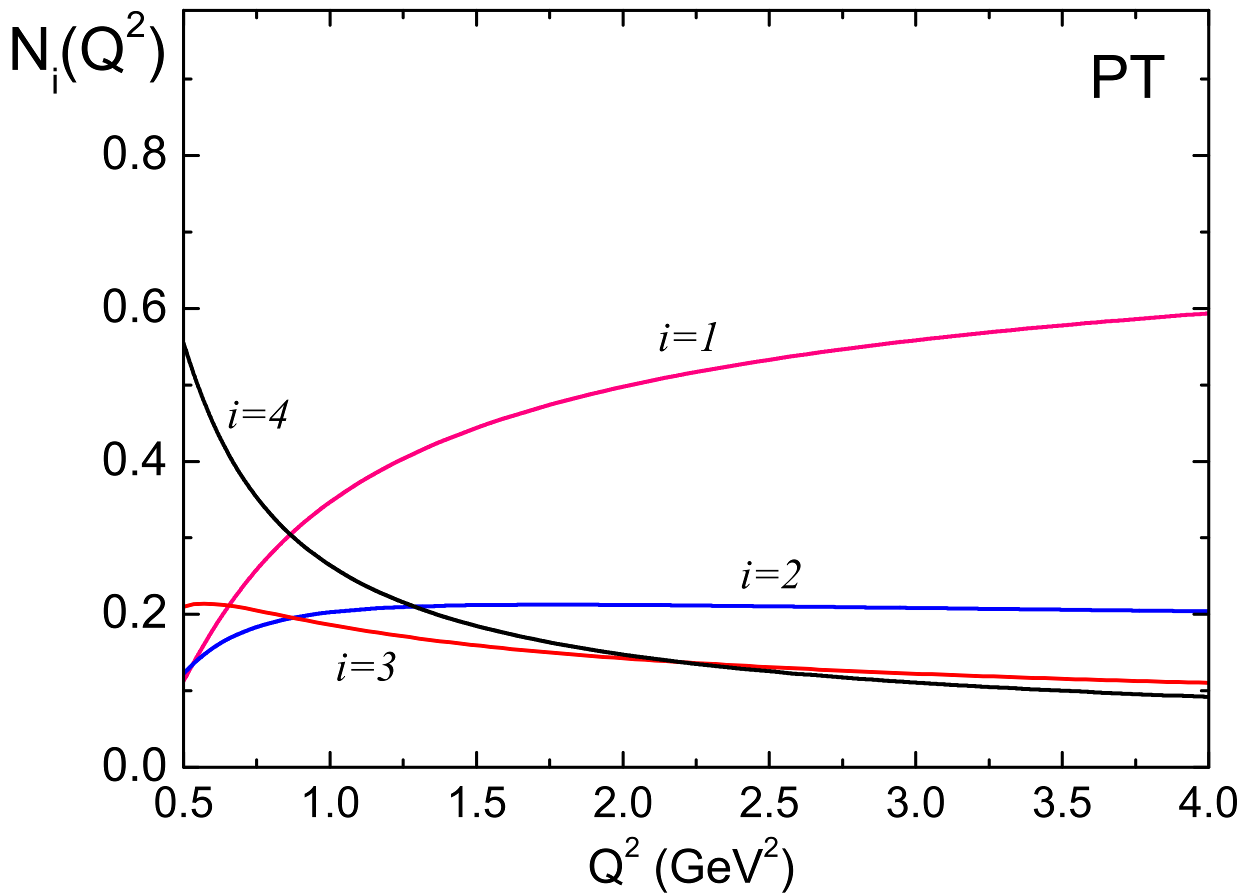
the minimal errors of pQCD contribution being rather big

$$\Delta(1.78) = 10\% \quad \textbf{and} \quad \Delta(1\,\mathrm{GeV}) = 36\%\,.$$

Even if, instead pQCD, we address to lattice simulation results, the menace will reduce quantitatively. But does not disappear.

of the PT series "blowing up" at $Q^2 \lesssim 2 - 3\,\mathrm{GeV}^2$ from

[Khandramaj, et al, hep-ph/1106.6352; Phys.Lett. B 706 (2012)]



Relative weight of 1-, 2-, 3-, 4-loop terms.

The singularity $e^{-1/g}$ is usual in Theory of Big Systems (representable via Functional or Path Integral) :

- **Turbulence**
- **Classic and Quantum Statistics**
- **Quantum Fields**

Reason : small parameter $g << 1$ at nonlinear structure

- **Energy Gap in SuperFluidity and SuperConductivity**
- **Tunneling in QM**
- **Quantum Fields (Dyson singularity), ...**

Generally, a certain AsymptSeries can correspond to a set of various functions.

. Their ”summation” is an Art.

The AS, singularity, factorials

- **The oldest argument [Dyson, 1949] on the QED singularity at** $\alpha = 0$

  relies on fictitious transformation

  $\alpha \to -\alpha \quad \sim \quad \mathrm{e} \to \pm \mathrm{i}\,\mathrm{e}$

  destroing hermiticity and unitarity.

- **An asymptotic** estimates for coefficients **of PT expansion for** $\mathrm{g}\varphi^4$ **, QED, QCD, have been obtained by** *steepest descent method for path integral*. **All they contain factorial.**

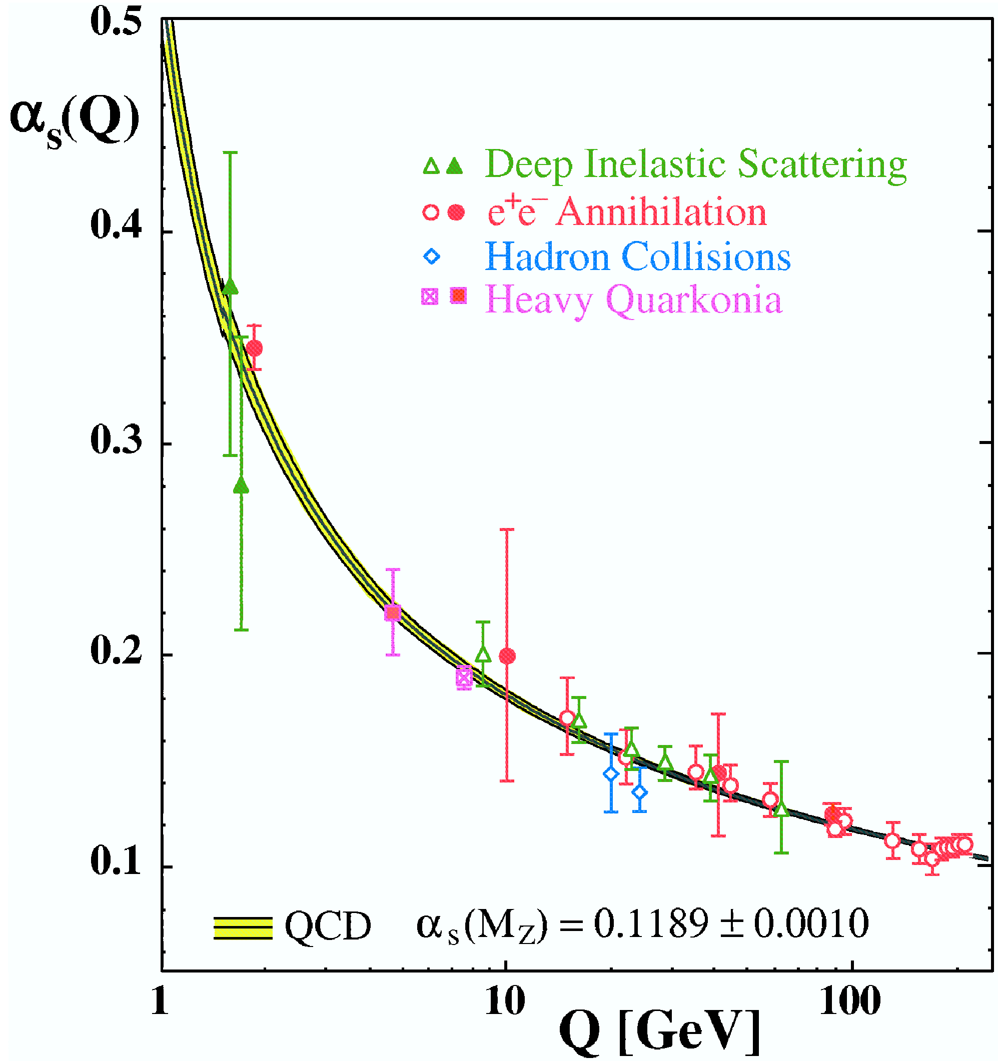
- **The type of singularity is** $e^{-1/g}$ **, the same for all the cases.**

- **The simple common reason is that by putting coupling to zero (**$g = 0,\ \alpha = 0$**) one** changes the type of equation.
  (E.g., changing non-linear eq. to linear one)

Dangerous domain for the pQCD

In QFT, all observables being renorm-invariant are expressible via RG-invariant coupling function; in perturb. QCD case – in the form of Taylor series in powers of strong "running" coupling $\alpha_s(Q)$. Due to non-abelian anti-screening, it decreases with the momentum-transfer Q increase (asymptotic freedom). Accordingly, $\alpha_s(Q)$ grows up to 0.3-0.4 values at $Q \sim 1 - 2\,GeV$ =

= Dangerous domain !



S.Bethke 2006 review

Perturb QCD contribution to Bjorken SR blows up

$$\Gamma_1(Q^2) = \frac{g_A}{6}\left[1 - \Delta^{PT}(Q^2)\right] + \Gamma_{HT}\,; \;, \qquad (3)$$

is known now up to the 4-loop term

$$\Delta^{PT} = \frac{\alpha_s(Q)}{\pi} + 0.363\alpha_s^2(Q) + 0.652\alpha_s^3(Q) + 1.804\,\alpha_s^4(Q) \qquad (4)$$

with the coefficient ratios close to the factorial [1 : 1 : 2 : 6] ones !

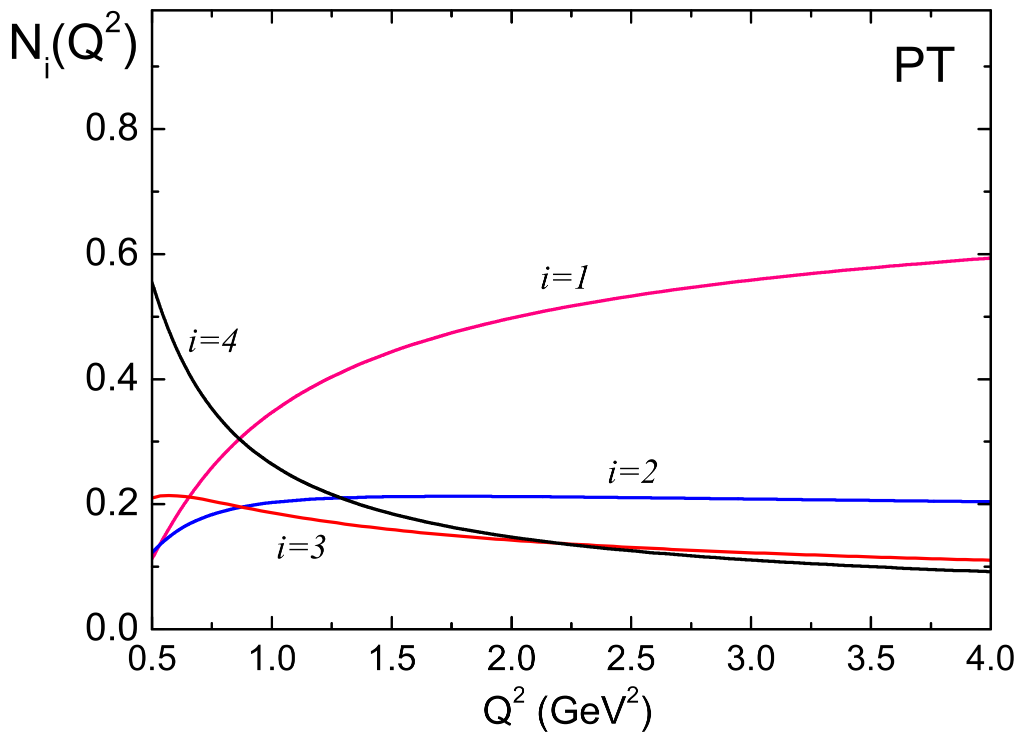
There are precise JLab data at very low Q values.

However, PT series

"blows up"

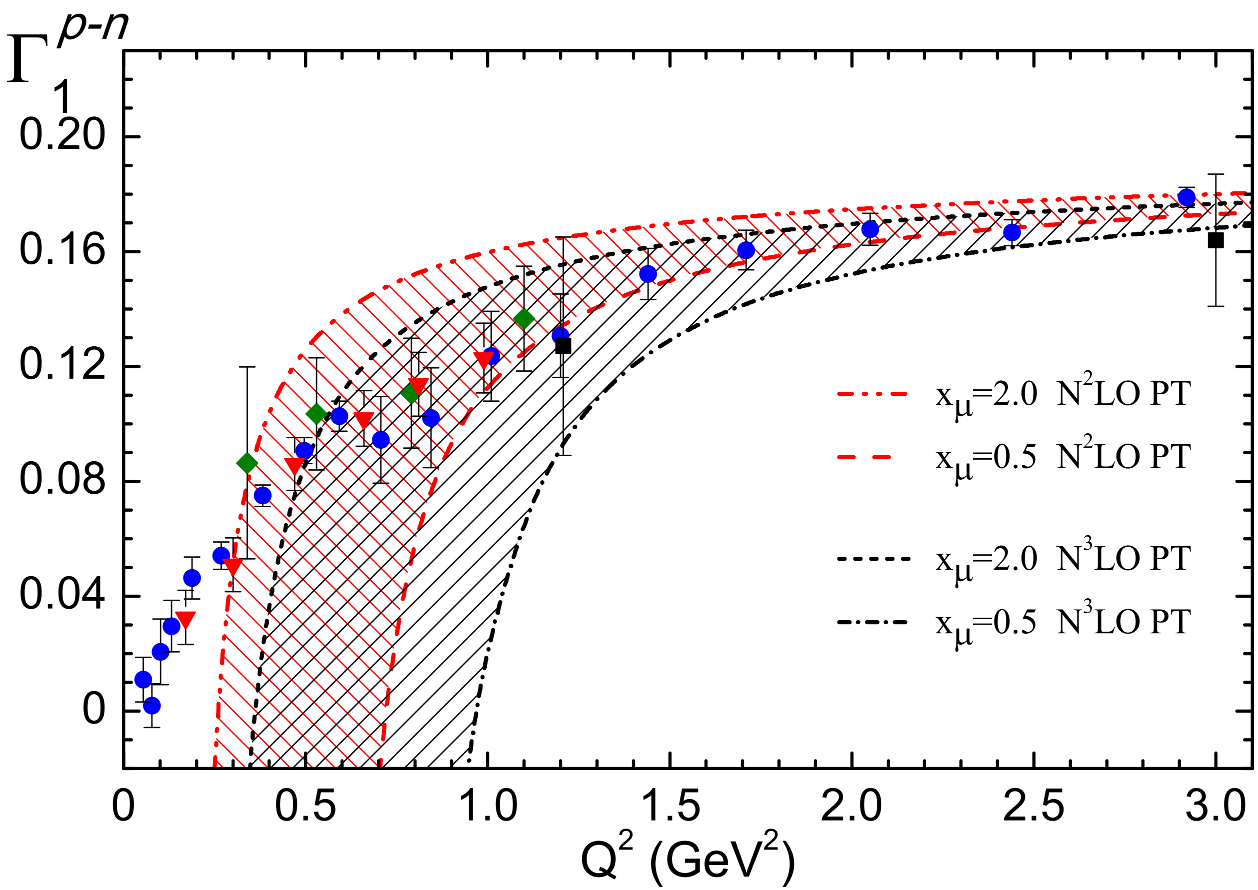
at $Q \lesssim 1.5 - 2\,\mathrm{GeV}$;

$\alpha_s(1.5) \sim 0.4; \quad \alpha_s(2) \sim 0.3$



Relative weight of 1-, 2-, 3-, 4-loop terms.

The 3- and 4-loop pQCD for Bjorken SumRule



The pQCD fit of JLab data for the 1st moment Γ_1

4-loop fit is slightly worse than the 3-loop

for detail address to [Khandramaj, et al., 2011]

Divergent Asymptotic Series

Illustration 1

$$A(g) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2-(g/4)\,x^4}\,dx\,; \quad g > 0\,. \qquad (5)$$

Expanding integrand in g and changing order of integration and summation one gets alternating divergent series

$$A(g) = \sum_{n\geq 0} (-g)^n A_n\,;\ \ A_n = \frac{2}{4^n\sqrt{\pi}\,n\,!} \int e^{-x^2} x^{4n}\,dx\,;\ \ A_0 = 1\,.(4)$$

The $n \to \infty$ limit for coefficients is pure factorial

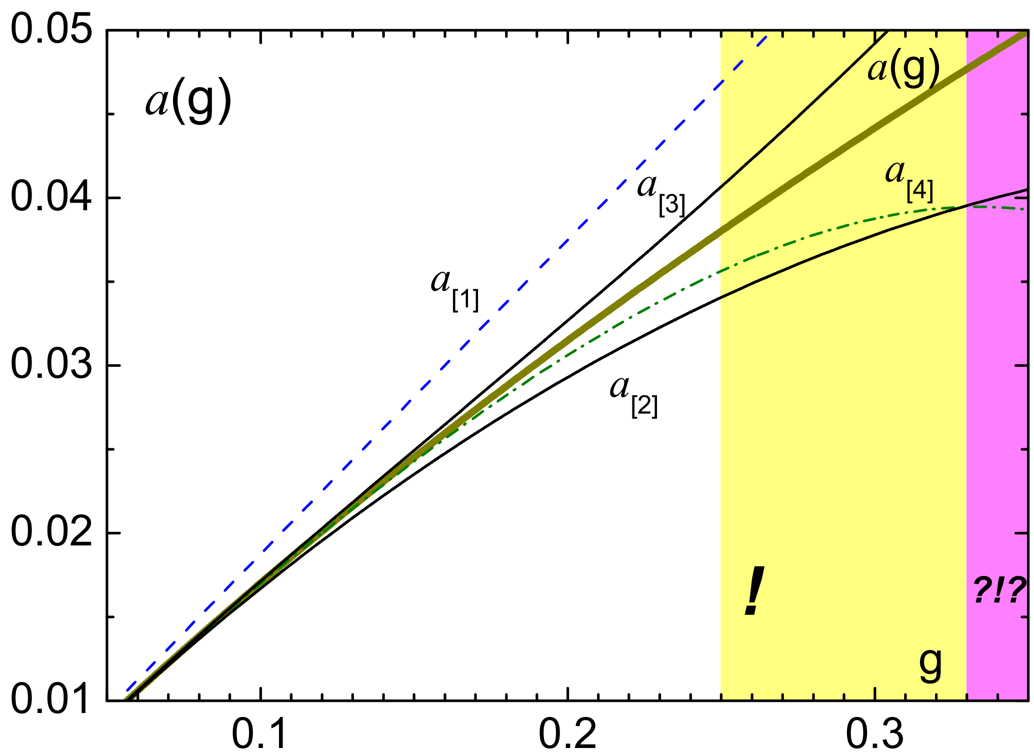
$$A_n = \left.\frac{\Gamma(2n+1/2)}{4^n\,\Gamma(n+1)}\right|_{n\gg 1} \to \quad A_n^{as} = \frac{\Gamma(n)}{\sqrt{2\,\pi}} = \frac{(n-1)\,!}{\sqrt{2\,\pi}}\,.$$

As it is known, function (3) has essential singularity at the origin $g = 0$ of the $e^{-1/g}$ type.

AsymSeries, Illustration 1, cont'd

The finite sums $a_{[n]}(g) = g\,A_1 - \cdots \pm A_n\,(-g)^n$ **of alternating series is compared with exact values of function (3)** $a(g) = 1 - A(g)$, **(solid curve)**

The exclamation mark "!" denotes beginning of yellow zone (caution light) = $a_{[4]}$ **is not better than** $a_{[3]}$ **while combination "?!?" marks the red zone, =** $a_{[4]}$ **is on the** $a_{[2]}$ **level.**



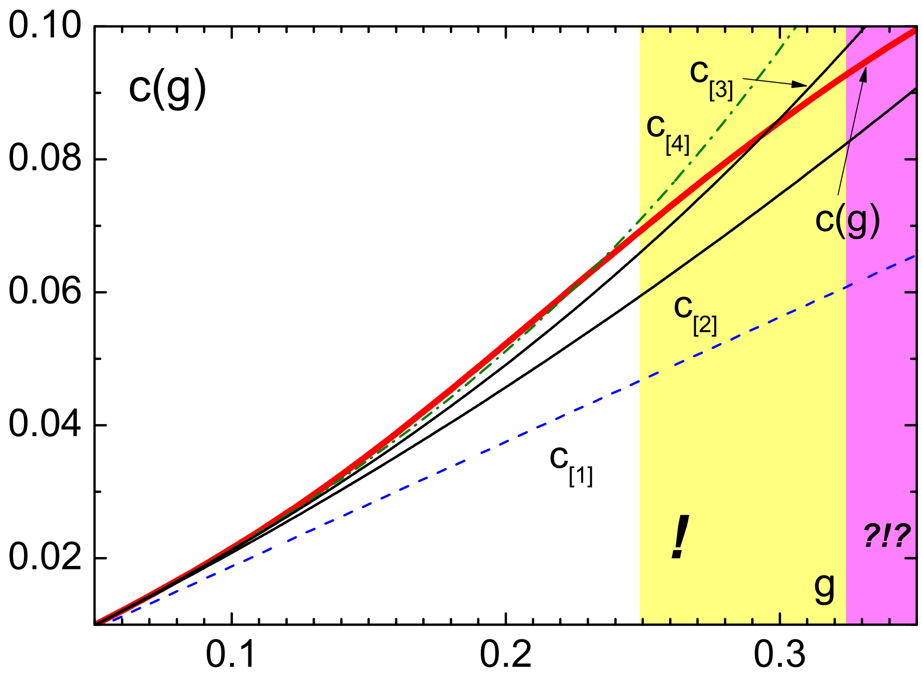
The $a_{[k]}$ approximants for function $A(g)$.

Another integral

$$C(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2(1-\frac{\sqrt{g}}{4}\,x)^2}\,dx \to \sum_k g^k C_k\,; \qquad C_k = A_k \qquad (6)$$

obeys non-alternating AS with <u>the same</u> coefficients.

Note that positions of the yellow and the red zones remain unchanged.
That nicely corresponds to the Poincaré estimate.
And correlates with the observed BjSR issue.



The $c_{[k]}$ approximants for $C(g)$.

Higher PT contributions to observables

Relative contributions (in %) of

1– , 2–, 3– and 4–loop terms

Process		**Scale/Gev**	***PT (in %)***			
the loop number =			**1**	**2**	**3**	**4**
Bjorken SR	**t**	**1**	**35**	**20**	**19**	**26**
Bjorken SR	**t**	**1.78**	**56**	**21**	**13**	**11**
GLS SumRule	**t**	**1.78**	**58**	**21**	**12**	**11**
Incl. τ-decay	**s**	**1.78**	**51**	**27**	**14**	**7**

Higher PT terms for $e^+e^- \to$ hadrons

Relative contributions of 1- ... 4–loop

terms in $e^+e^- \to$ hadrons

Function	Scale/Gev	PT terms (in %)				Comment
r(s)	1	65	19	55	– 39	?!?
r(s)	1.78	73	13	24	-10	? !
d(Q)	1	56	17	11	16	in agenda
d(Q)	1.78	75	14	6	5	in agenda

In the $r(s)$ higher coefficients –

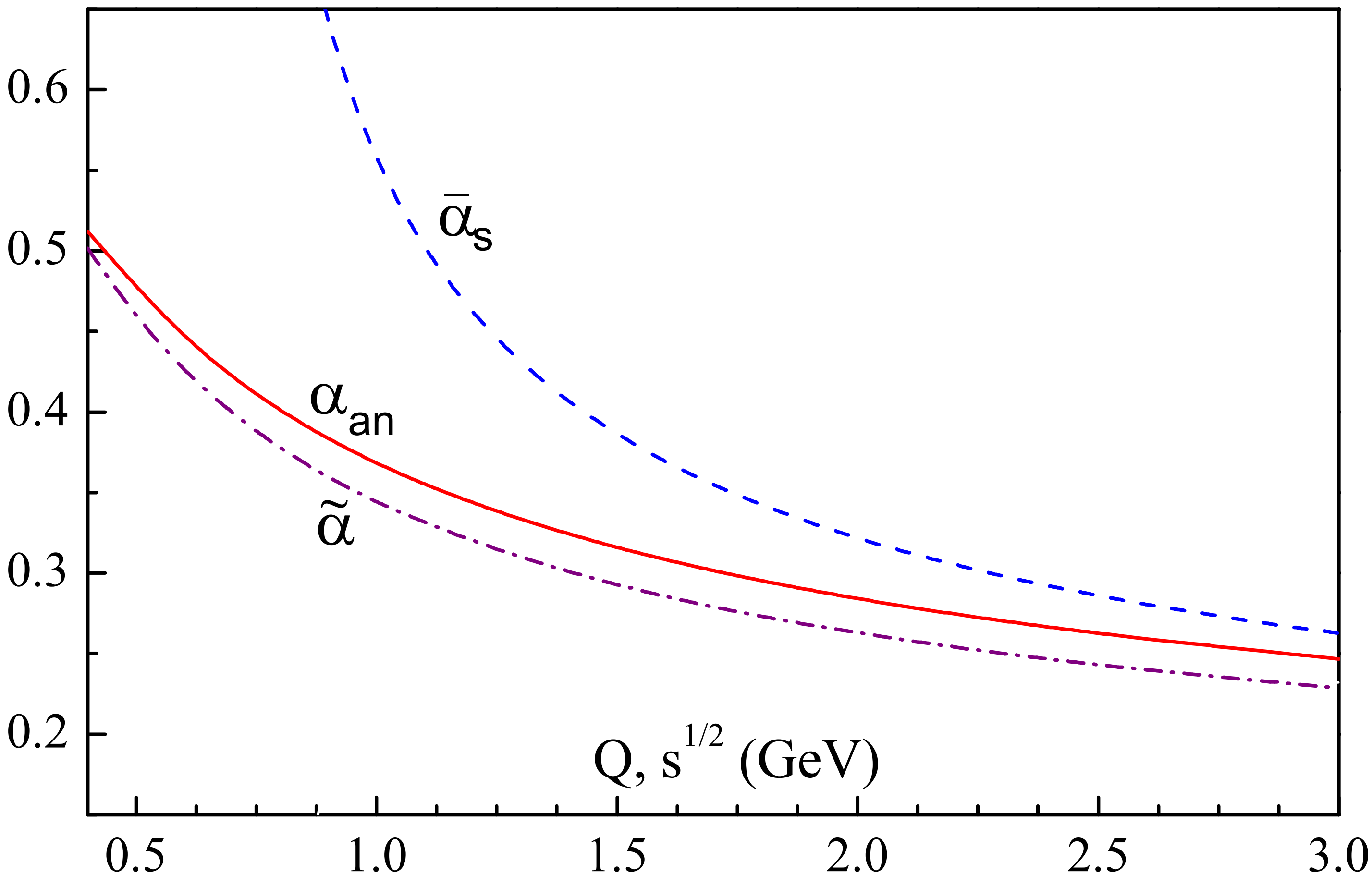
— terrible effect of the π^2 terms !

Few words on the APT

- **Non-power set of PT-expansion functions $\mathcal{A}_k(Q)$ instead of the $\alpha_s(Q)$ powers ;**
- **All the functions reflect RG-invariance and causality via Qr-analyticity;**
- **Euclidean $\mathcal{A}_k$ expansion functions are different from the Minkowskian $\mathfrak{A}_k$ ones ; all of them :**
  - **are related via differential recurrent relations**
  - **the higher functions $k \geq 2$ vanish at the IR limit ;**
  - **in the region above 1-2 GeV quickly tend to the α_s powers ;**
- **As all the expansion functions incorporate e^{-1/α_s} structures, the PT convergence improves drastically ;**

Numerous applications to data analysis demonstrate the APT effectiveness in the 1 GeV region.
However, below 500 MeV the APT meets some troubles.

Comparing APT couplings with $\alpha_s(Q^2)$



Red curve – $\alpha_{an} = \alpha_{APT}(Q)$, [black dash-dotted – $\tilde{\alpha}_{APT}(\sqrt{s})$]