T-duality in coordinate dependent background

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Outline

- We consider the closed string propagating in the weakly curved background which consists of constant metric and Kalb-Ramond field with infinitesimally small coordinate dependent part.
- We perform T-duality transformations along coordinates on which the Kalb-Ramond field depends.
- We obtain the T-dual theory defined in the non-geometric double space, described by the Lagrange multiplier y_μ and its T-dual in absence of background fields $\tilde{y}_\mu$.
- We find the global symmetry of the T-dual theory, in the doubled target space and the procedure to return to the initial theory.
- Global issues

Bosonic string in the weakly curved background

- Action for the closed string in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$ equals

$$S[x] = \kappa \int_{\Sigma} d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma,$$

  where

$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x].$$

  - $G_{\mu\nu} = G_{\nu\mu}$ metric tensor
  - $B_{\mu\nu} = -B_{\nu\mu}$ Kalb-Ramond field

Weakly curved background

- consistency of the theory implies conformal invariance on the quantum level
- space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0$$

$$D_\rho B^\rho{}_{\mu\nu} = 0$$

- particular solution

$$G_{\mu\nu}[x] = const$$

$$B_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho$$

$$b_{\mu\nu} = const, \quad B_{\mu\nu\rho} = const$$

- we choose infinitesimal $B_{\mu\nu\rho}$ and we work up to the terms linear in it

Standard Bouscher's construction

- The premise is that the target space has isometries.
- It is possible to chose adopted coordinates $x^\mu = (x^i, x^a)$, so that the isometries act as translations of x^a components.
- If background fields are x^a-independent, the action is invariant under the global shift symmetry.

Generalized Bouscher's construction

- In the weakly curved background, despite of x^a-dependence of the background fields, the global shift $\delta x^\mu = \lambda^\mu = const$, leaves the action for the closed string, invariant.
- For simplicity we suppose that all the coordinates are compact. As $B_{\mu\nu}$ is linear in coordinate, one has

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho}\lambda^\rho \int d^2\xi \partial_+ x^\mu \partial_- x^\nu = \frac{\kappa}{3} B_{\mu\nu\rho}\lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi \partial_\alpha x^\mu \partial_\beta x^\nu .$$

  This is proportional to the total divergence

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho}\lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi \partial_\alpha (x^\mu \partial_\beta x^\nu) = 0,$$

  which vanishes in the case of
  - the closed string and
  - the topologically trivial mapping of the world-sheet into the space-time.

Gauging the symmetry

- To localize this global symmetry, we introduce the world-sheet gauge fields v_α^μ and substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu.$$

- We want the covariant derivatives to be gauge invariant, so we impose the transformation law for the gauge fields

$$\delta v_\alpha^\mu = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau,\sigma)).$$

- This replacement is however not sufficient to make the action locally invariant because the background field $B_{\mu\nu}$ in the weakly curved background, depends on the coordinate x^μ which is not gauge invariant.

Invariant coordinate

- The background field $B_{\mu\nu}$ depends on the coordinate x^μ which is not gauge invariant.
- To make the action invariant we should replace the coordinate x^μ, with some extension for it, where only already introduced gauge fields v^μ_α will appear.
- We take

$$x^\mu_{inv} = x^\mu + V^\mu[v_+, v_-].$$

- V^μ is line integral of the gauge fields

$$V^\mu[v_+, v_-] \equiv \int_P d\xi^\alpha v^\mu_\alpha = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-),$$

taken along the path P, from the initial point $\xi^\alpha_0(\tau_0, \sigma_0)$ to the final one $\xi^\alpha(\tau, \sigma)$.

Gauged action

- Dual theory should be equivalent to the initial one.
- New degrees of freedom should not be introduced through the gauge fields.
- Therefore, we require the corresponding field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha,$$

to vanish.
- This is achieved by introducing the Lagrange multiplier y_μ, and the appropriate term in the Lagrangian which forces $F^\mu_{+-} \equiv \partial_+ v^\mu_- - \partial_- v^\mu_+ = -2F^\mu_{01}$ to vanish.
- The proposition for the gauge invariant action is

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}[x_{inv}] D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big],$$

where the last term is equal $\frac{1}{2} y_\mu F^\mu_{+-}$ up to the total divergence.

Gauge fixed action

- We fix the gauge with $x^\mu = 0$.
- The gauge fixed action equals

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu}[V] v_-^\nu + \frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu) \Big],$$

where y_μ and $v_\pm^\mu$ are independent variables and

$$V^\mu[v_+, v_-] \equiv \int_P d\xi^\alpha v_\alpha^\mu = \int_P (d\xi^+ v_+^\mu + d\xi^- v_-^\mu).$$

T-dual action

- The T-dual action will be obtained by integrating out the gauge fields from the gauge fixed action.
- The equations of motion with respect to the gauge fields $v_\pm^\mu$ are

$$\Pi_{\mp\mu\nu}[V]v_\pm^\nu + \frac{1}{2}\partial_\pm y_\mu = \mp\beta_\mu^\mp[V].$$

- $\beta_\mu^\mp$ comes from the variation with respect to the argument of the background fields, and equals $\beta_\mu^\alpha[V] \equiv \partial_\mu B_{\nu\rho}\epsilon^{\alpha\beta}V^\nu\partial_\beta V^\rho$.
- EM can be rewritten as

$$v_\pm^\mu(y) \quad = \quad -\kappa\,\Theta_\pm^{\mu\nu}[V(y)]\Big[\partial_\pm y_\nu \pm 2\beta_\nu^\mp[V(y)]\Big].$$

$$\Theta_\pm^{\mu\nu}[V] = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu}[V] \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}[V],$$

and $G^E_{\mu\nu} \equiv [G - 4BG^{-1}B]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the non-commutativity parameter and the effective metric.

T-dual action

- Substituting equation of motion into the gauge fixed action, we obtain T-dual action

$$^\star S[y] \equiv S_{fix}[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_-^{\mu\nu}[V(y)]\partial_- y_\nu ,$$

where we neglected the term $\beta_\mu^- \beta_\nu^+$ as the infinitesimal of the second order.

- To obtain the explicit expression for the action we should substitute the solution for $v_\pm^\mu$ and V^μ.
- In the general case, these can not be trivially found.

Solving EM iteratively

- We will solve equations of motion iteratively, separately addressing the Minkowski background and the flat background (zeroth order iteration) case. At each step, we will find the explicit expression for $v_\pm^\mu$ and V^μ.
- Minkowski background fields (T_0-duality)
  - background $G_{\mu\nu} \to \eta_{\mu\nu}, B_{\mu\nu} \to 0$
  - the gauge fields $v_\pm^{(0)\mu} = \pm(G^{-1})^{\mu\nu}\partial_\pm y_\nu$
  - The T-dual action ${}^\star S[y] = \frac{\kappa}{2}\int d^2\xi\ (G^{-1})^{\mu\nu}\partial_+ y_\mu \partial_- y_\nu$.
  - EM $\partial_+\partial_- y_\mu = 0$, with the solution $y_\mu = y_{+\mu}(\xi^+) + y_{-\mu}(\xi^-)$.
  - Interpretation for V^μ

$$
\begin{aligned}
V^{(0)\mu} &= (G^{-1})^{\mu\nu}\int_P (d\xi^+\partial_+ y_\nu - d\xi^-\partial_- y_\nu) \\
&= (G^{-1})^{\mu\nu}\int_P (d\tau y'_\nu + d\sigma \dot{y}_\nu) = (G^{-1})^{\mu\nu}\tilde{y}_\nu,
\end{aligned}
$$

is the T_0-dual coordinate and $\tilde{y}_\mu = y_{+\mu}(\xi^+) - y_{-\mu}(\xi^-)$.

Solving EM iteratively

- The flat background (T_1-duality)
  - the constant background $G_{\mu\nu}[x] \to G_{\mu\nu}, \quad B_{\mu\nu}[x] \to b_{\mu\nu}$
  - the quantities $G^E_{\mu\nu}[x]$, $\theta^{\mu\nu}[x]$, $\Pi_{\pm\mu\nu}[x]$, $\Theta^{\mu\nu}_{\pm}[x]$ reduce to their constant parts

$$G^E_{\mu\nu}[x] \to g_{\mu\nu} \equiv [G - 4bG^{-1}b]_{\mu\nu},$$
$$\theta^{\mu\nu}[x] \to \theta_0^{\mu\nu} \equiv -\frac{2}{\kappa}[g^{-1}bG^{-1}]^{\mu\nu},$$
$$\Pi_{\pm\mu\nu}[x] \to \Pi_{0\pm\mu\nu} \equiv b_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu},$$
$$\Theta^{\mu\nu}_{\pm}[x] \to \Theta^{\mu\nu}_{0\pm} \equiv \theta_0^{\mu\nu} \mp \frac{1}{\kappa}(g^{-1})^{\mu\nu}.$$

  - $\Pi_{0+\mu\nu}$ is constant, $\beta^{\pm}_{\mu}$ vanishes and gauge fields equal

$$v^{(1)\mu}_{\pm} = -\kappa\,\Theta^{\mu\nu}_{0\pm}\partial_{\pm}y_{\nu}.$$

Solving EM iteratively

- The flat background (T_1-duality)
  - The T_1-dual action $S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_{0-}^{\mu\nu} \partial_- y_\nu$, has in comparison to T_0 case the additional term depending on the constant antisymmetric background field $b_{\mu\nu}$. Because this term is topological, it does not contribute to the equations of motion and therefore these are the same as in the T_0-case.
  - $V^{(1)\mu}$ becomes

$$
\begin{aligned}
V^{(1)\mu} &= -\kappa \Theta_{0+}^{\mu\nu} y_{+\nu}(\xi^+) - \kappa \Theta_{0-}^{\mu\nu} y_{-\nu}(\xi^-) \\
&= (g^{-1})^{\mu\nu} [(2bG^{-1})_\nu{}^\rho y_\rho + \tilde{y}_\nu].
\end{aligned}
$$

T dual action in the weakly curved background

- $V^{(1)\mu} = (g^{-1})^{\mu\nu}[(2bG^{-1})_\nu{}^\rho y_\rho + \tilde{y}_\nu]$
- gauge fields $v_\pm^\mu = -\kappa\Theta_\pm^{\mu\nu}[V^{(1)}]\Big[\partial_\pm y_\nu \pm \beta_\nu^\mp[V^{(1)}]\Big]$
- The T-dual action $^\star S[y] = \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta_-^{\mu\nu}[V^{(1)}]\partial_- y_\nu$
- Initial action transforms into the T-dual action under

$$\partial_\pm x^\mu \to \partial_\pm y_\mu, \qquad \Pi_{+\mu\nu}[x] \to \frac{\kappa}{2}\Theta_-^{\mu\nu}[V^{(1)}]$$

$$\begin{aligned} G_{\mu\nu} &\to {}^\star G^{\mu\nu}[y,\tilde{y}] = (G_E^{-1})^{\mu\nu}[V^{(1)}] \\ B_{\mu\nu}[x] &\to {}^\star B^{\mu\nu}[y,\tilde{y}] = \frac{\kappa}{2}\theta^{\mu\nu}[V^{(1)}] \end{aligned}$$

- Dual background fields $^\star G^{\mu\nu}$ and $^\star B^{\mu\nu}$ are defined on the doubled target space $(y,\tilde{y})$.

The T-dual of the T-dual theory

- T-dual theory is by construction physically equivalent to the initial one.
- We should expect that the T-dual of the T-dual theory is just the initial theory.
- The shift $\delta y_\mu = \lambda_\mu$ is not the global symmetry.
- Let us find the global symmetry of the T-dual action.
- We consider the global transformations which differ for the different chirality parts $\delta y_{\pm\mu} = \mp \Pi_{0\mp\mu\nu}\lambda^\nu$, and are chosen in such a way that V^μ is globally invariant $\delta V^\mu = 0$.
- Let us localize this symmetry and find the corresponding locally invariant action. We are going to gauge independently both chirality components with the chiral local parameters

$$\delta y_{\pm\mu} = \mp \Pi_{0\mp\mu\nu}\lambda^\nu_\pm(\xi^\pm).$$

Gauging the symmetry

- We covariantize the derivatives introducing the gauge fields $u_{\pm\mu}$

$$D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu}.$$

- Demanding

$$\delta D_\pm y_\mu = 0,$$

  we require that $u_{\pm\mu}$ transform as

$$\delta u_{\pm\mu} = \pm\Pi_{0\mp\mu\nu}\lambda^\nu_\pm(\xi^\pm).$$

- How does V^μ transform? Using $\Theta^{\mu\nu}_{0\pm}\Pi_{0\mp\nu\rho} = \frac{1}{2\kappa}\delta^\mu_\rho$, we conclude

$$\delta V^\mu = \frac{1}{2}(\lambda^\mu_+ - \lambda^\mu_-) = \frac{1}{2}\tilde{\lambda}^\mu.$$

Action dual to T-dual action

- To construct an invariant expression $V^\mu_{inv} = V^\mu + U^\mu[u_+, u_-]$, we will take U^μ in a form

$$U^\mu = -\kappa\Theta^{\mu\nu}_{0+}\int d\xi^+ u_{+\nu} - \kappa\Theta^{\mu\nu}_{0-}\int d\xi^- u_{-\nu},$$

which produces

$$\delta V^\mu_{inv} = 0.$$

- The dual invariant action is

$$\begin{aligned} {}^\star S_{inv} &= \frac{\kappa^2}{2}\int d^2\xi D_+ y_\mu \Theta^{\mu\nu}_-[V+U] D_- y_\nu \\ &+ \frac{\kappa}{2}\int d^2\xi(u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu), \end{aligned}$$

where z^μ is Lagrange multiplier and the second term makes the gauge fields $u_{\pm\mu}$ nonphysical.

Action dual to T-dual action

- The gauge fixing $y_{+\mu} = y_{-\mu} = 0$ produces $D_\pm y_{\pm\mu} = u_{\pm\mu}$ and $V^\mu = 0$, so the action becomes

$$\begin{aligned} {}^\star S_{fix}[z, u_\pm] &= \frac{\kappa^2}{2} \int d^2\xi\, u_{+\mu} \Theta_-^{\mu\nu}[U] u_{-\nu} \\ &+ \frac{\kappa}{2} \int d^2\xi (u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu). \end{aligned}$$

- The equations of motion with respect to z^μ

$$\partial_+ u_{-\mu} - \partial_- u_{+\mu} = 0,$$

  have the solution $u_{\pm\mu} = \partial_\pm y_\mu$,
- Therefore $U^\mu = V^\mu$
- The action becomes

$${}^\star S_{fix}[u_\pm = \partial_\pm y] = \frac{\kappa^2}{2} \int d^2\xi\, \partial_+ y_\mu \Theta_-^{\mu\nu}[V] \partial_- y_\nu.$$

Integrating out the gauge fields

- By varying the action with respect to the gauge fields $u_{\pm\mu}$, we obtain the equations of motion

$$\partial_\pm z^\mu = -\kappa\Theta_\pm^{\mu\nu}[U]\Big[u_{\pm\nu} \pm 2\beta_\nu^\mp[U]\Big].$$

- Using the expression $\Theta_\pm^{\mu\nu}\Pi_{\mp\nu\rho} = \frac{1}{2\kappa}\delta_\rho^\mu$, we can extract $u_{\pm\mu}$

$$u_{\pm\mu} = -2\Pi_{\mp\mu\nu}[U]\partial_\pm z^\nu \mp 2\beta_\mu^\mp[U].$$

- We solve this equation iteratively.
- The zeroth order values

$$U^\mu = z^\mu,$$
$$\beta_\mu^\pm[U] = \beta_\mu^\pm[z]$$

- The solution is $u_{\pm\mu} = -2\Pi_{\mp\mu\nu}[z]\partial_\pm z^\nu \mp 2\beta_\mu^\mp[z]$
- T-dual of the T-dual action
  ${}^{\star\star}S[z] \equiv {}^\star S_{fix}[z] = \kappa\int d^2\xi\partial_+ z^\mu\Pi_{+\mu\nu}[z]\partial_- z^\nu$

T-dual transformations

- T-dual transformation of the variables law

$$\partial_\pm x^\mu \cong -\kappa\Theta_\pm^{\mu\nu}[g^{-1}(2by+\tilde{y})]\Big[\partial_\pm y_\nu \pm \beta_\nu^\mp[g^{-1}(2by+\tilde{y})]\Big]$$

- and its inverse

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}[z]\partial_\pm z^\nu \mp 2\beta_\mu^\mp[z]$$

Features of T-duality

T-duality interchanges momentum and winding numbers, equations of motion and Bianchi identities.

Original theory S	$\longrightarrow$	T-dual theory $^\star S$
Noether current j_μ^α		Topological current $^\star i_\mu^\alpha = -\kappa\epsilon^{\alpha\beta}\partial_\beta y_\mu$
Conservation law = Equation of motion		Conservation law = Bianchi identity
$\partial_\alpha j_\mu^\alpha = 0$		$\partial_\alpha {}^\star i_\mu^\alpha = 0$
Noether conserved charge		Topological conserved charge
$Q_\mu = \int_{-\pi}^{\pi} d\sigma j_\mu^0 = \int_{-\pi}^{\pi} d\sigma \pi_\mu = P_\mu$		$^\star q_\mu = \int_{-\pi}^{\pi} d\sigma \, {}^\star i_\mu^0 = \kappa \int_{-\pi}^{\pi} d\sigma y'_\mu = {}^\star W_\mu$
T-dual of T-dual theory $^{\star\star}S = S$	$\longleftarrow$	T-dual theory $^\star S$
Topological current $i^{\alpha\mu} = -\kappa\epsilon^{\alpha\beta}\partial_\beta x^\mu$		Noether current $^\star j^{\alpha\mu}$
Conservation law = Bianchi identity		Conservation law = Equation of motion
$\partial_\alpha i^{\alpha\mu} = 0$		$\partial_\alpha {}^\star j^{\alpha\mu} = 0$
Topological conserved charge		Noether conserved charge
$q^\mu = \int_{-\pi}^{\pi} d\sigma \, i^{0\mu} = \kappa \int_{-\pi}^{\pi} d\sigma x'^\mu = W^\mu$		$^\star Q^\mu = \int_{-\pi}^{\pi} d\sigma \, {}^\star j^{0\mu} = \int_{-\pi}^{\pi} d\sigma \, {}^\star \pi^\mu = {}^\star P^\mu$

Quantum level - global issues

- Problems
  - x_{inv} and V are multivalued
  - $S_{fix} \to S$ global issues problem
  - higher genus in quantum theory
- Solution in quantum theory
  - Euclidian path integral (using differential form notation) $Z = \sum_{g=0}^{\infty} \int \mathcal{D}y\mathcal{D}v \, e^{-\frac{\kappa}{2}\int vG\star v + i\kappa \int vB(V)v + \frac{i\kappa}{2}\int vdy}$.
  - Hodge decomposition of the 1-form v into exact, co-exact and the harmonic parts
    - $v = dx + d^\dagger v_{ce} + v_h$
    - $dy = dy_e + y_h$ (there is no co-exact part because $ddy = 0$)
    - d is exterior derivative, $d^\dagger$ is the adjoint to d with respect to the inner product $(da, b) = (a, d^\dagger b)$
  - All nontrivial holonomies come from the harmonic parts
  - Integration by parts in terms of y_e is allowed

Quantum level - global issues

- $\mathcal{D}y \to \mathcal{D}y_e \sum_{H_y}$ path integration over local degrees of freedom and sum over topologies (from y_h)
- $\mathcal{D}v \to \mathcal{D}x\mathcal{D}d^\dagger v_{ce}\, dH_v$
- $\int \mathcal{D}y_e e^{\frac{i\kappa}{2}\int v(dy_e+y_h)} = e^{\frac{i\kappa}{2}\int vy_h}\delta(dv)$
- $\delta(dv) \to dv = 0 \to dd^\dagger v_{ce} = 0 \to d^\dagger v_{ce} = 0 \to \delta(d^\dagger v_{ce})$
- $Z = \int \mathcal{D}xdH_v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma vy_h}$
- $v = dx + v_h$
- Riemann bilinear relation for closed forms $\int_\Sigma vy_h = \sum_{i=1}^{g}\left[\oint_{a_i} v \oint_{b_i} y_h - \oint_{a_i} y_h \oint_{b_i} v\right]$
- $a_i, b_i\ i = 1, 2, \cdots, g$ canonical homology basis

Quantum level - global issues

- Lagrange multipliers y-periodic $\oint_{a_i} y_h = n_{a_i}$, $\oint_{b_i} y_h = n_{b_i}$
- n_{a_i} and n_{b_i} are the winding numbers around cycles a_i, b_i
- Periods of 1-form v $\oint_{a_i} v = A_i$, $\oint_{b_i} v = B_i$
- $\int_\Sigma v y_h = \sum_{i=1}^{g} (n_{b_i} A_i - n_{a_i} B_i)$
- $H_y \to n_{a_i}, n_{b_i}$, $dH_v \to dA_i dB_i$
- $\sum_{n_{a_i}, m_{b_i} \in Z} e^{\frac{i\kappa}{2} \sum_{i=1}^{g} (n_{b_i} A_i - n_{a_i} B_i)} = \delta(A_i)\delta(B_i)$ periodic delta functions
- $Z = \int \mathcal{D}x \, dA_i dB_i \delta(A_i)\delta(B_i) \, e^{-\frac{\kappa}{2} \int_\Sigma v \, G \, {}^\star v + i\kappa \int_\Sigma v B[V] v}$.

V-dependence on the path P

- $v = dx + v_h \to V = x - x_0 + \int_P v_h$
- $V^\mu[P] - V^\mu[P_1] = \oint_{PP_1^{-1}} v_h^\mu$ depends only on the cohomology class of v_h
- If PP_1^{-1} is homological to a curve $\sum_i \left[n_i a_i + m_i b_i\right]$ then $V^\mu[P] - V^\mu[P_1] = \sum_i \left[n_i A_i + m_i B_i\right]$
- $dA_i dB_i \to A_i = 0 = B_i \to V(P) = V(P_1)$ is single valued $v = dx \to V = x$
- The initial action is regained

$$
\begin{aligned}
Z &= \int \mathcal{D}x e^{-\frac{\kappa}{2}\int_\Sigma dx\, G\, {}^\star dx + i\kappa \int_\Sigma dx B[x] dx} \\
&= \int \mathcal{D}x e^{-\kappa \int_\Sigma d^2\xi \partial x \Pi_+[x] \bar{\partial} x}
\end{aligned}
$$

- The winding modes of the Lagrange multiplier $y^\mu\left(n_{a_i}, n_{b_i}\right)$ act as the Lagrange multipliers for the holonomies