Parastatistics and C_∞ Algebras

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we used the shorthand $\partial m_n := dm_n - (-1)^n (m_n(d, id, \ldots, id) + m_n(id, d, id, \ldots, id) + \ldots + m_n(id, \ldots, id, d))$

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Let A be a DGA and let $H^\bullet(A)$ be the cohomology ring of A. There is an A_∞- algebra structure on $H^\bullet(A)$ with $m_1 = 0$ and m_2 induced by the multiplication on A, constructed from the DGA A, such that there is a quasi-isomorphism of A_∞-algebras $H^\bullet(A) \xrightarrow{\iota} A$ lifting the identity of $H^\bullet(A)$.

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Corollary ("Metaphors" of the $\wedge$-product)

The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H^\bullet(\Lambda^p \mathfrak{g}^*, \delta^p)$ is a commutative A_∞-algebra, or C_∞-algebra.

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Cohomology

$$E_n^* \cong \mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K})$$

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Abelian Lie alg and Koszul Complex of $S(V)$

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The Yoneda algebra $\mathrm{Ext}^{\bullet}_{S(v)}(\mathbb{K}, \mathbb{K}) \cong \Lambda^{\bullet} V^* \cong S(V)^!$

Green Parastatistics Algebra and its Fock Space

Creation-Annihilation Algebra of H.S.Green (1953)

$$
\begin{aligned}
[[a_i^\dagger, a_j], a_k^\dagger] &= 2\delta_{jk} a_i^\dagger \qquad & [[a_i^\dagger, a_j], a_k] &= -2\delta_{ik} a_j \\
[[a_i^\dagger, a_j^\dagger], a_k^\dagger] &= 0 \qquad & [[a_i, a_j], a_k] &= 0
\end{aligned}
\tag{6}
$$

Green Parastatistics Algebra and its Fock Space

Creation-Annihilation Algebra of H.S.Green (1953)

$$\begin{aligned} [[a_i^\dagger, a_j], a_k^\dagger] &= 2\delta_{jk} a_i^\dagger & \qquad [[a_i^\dagger, a_j], a_k] &= -2\delta_{ik} a_j \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] &= 0 & \qquad [[a_i, a_j], a_k] &= 0 \end{aligned} \tag{6}$$

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Basis-free definition of $\mathcal{F}$ generated in $V = \bigoplus_{i \in I} \mathbb{K} a_i^\dagger$

$$PS(V) = T(V)/([[V, V]_\otimes, V]_\otimes)$$

where $(\mathfrak{I})$ stands for a two-sided ideal generated by
$\mathfrak{I} = \{[[a_i^\dagger, a_j^\dagger], a_k^\dagger]\} \subset V^{\otimes 3}$

Free 2-Step Nilpotent Lie algebra and $PS(V)$

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The algebra $PS(V)$ is the Universal Enveloping Algebra of the graded Lie algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

$$PS(V) := U\mathfrak{g} = U(V \oplus \Lambda^2 V) \tag{7}$$

Parastatistics Algebra $PS(V)$ and Young Tableaux

Theorem

Let $S^{\lambda}(V)$ be the Schur module associated with Young diagram λ. The algebra $PS(V)$ is a $GL(V)$-model, i.e., every irreducible polynomial $GL(V)$-representations appears once and exactly once

$$PS(V) = \bigoplus_{\lambda} S^{\lambda}(V)$$

The Young tableaux label the basis of $PS(V)$.

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Proof: The Cauchy formula is an identity of characters

$$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_\lambda s_\lambda(x)$$

where $s_\lambda(x) = chS^\lambda(V)$ stands for the Schur polynomial of dim V variables.

Chevalley-Eilenberg for 2-nilpotent $\mathfrak{g} = V \oplus \Lambda^2 V$

The Poincaré-Birkhoff-Witt theorem for $PS(V) := U\mathfrak{g}$ yields

$$PS(V) \cong S(V \oplus \Lambda^2 V) \tag{8}$$

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Homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ is bigraded

$$H_\bullet(\mathfrak{g}, \mathbb{K}) \cong H_\bullet(\mathbb{K} \otimes_{PS} C(\mathfrak{g})) \cong \mathrm{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K})$$

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Homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ as a $GL(V)$-module

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Theorem (Jozefiak and Weyman)

The homology of the chain complex $(\Lambda^p \mathfrak{g}, \partial_p)$ decomposes into irreducible $GL(V)$-modules as follows

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\Lambda \mathfrak{g}, \partial) \cong \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda : \lambda = \lambda'} S^\lambda(V) \quad (13)$$

the sum is over self-conjugate λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

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The minimal free resolution of $\mathbb{K}$ by left PS-modules

$$\mathbf{P}: \qquad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K}$$

$P_n = PS \otimes E_n$ with $E_n = \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$

Representation theory meaning of Littlewood formula

Jozefiak and Weyman(1985)
Representation-theory interpretation of Littlewood formula

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Corollary ("Metaphors" of the $\wedge$-product)

The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong H^\bullet(\Lambda^p \mathfrak{g}^*, \delta^p) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ is a commutative homotopy algebra, or C_∞-algebra.

Cohomology $H^\bullet(\mathfrak{g},\mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K})$ as a C_∞ algebra

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Proof.

Apply the Kadeishvili Homotopy transfer theorem to the commutative DG algebra corresponding to $\mathfrak{g} = V \oplus \Lambda^2 V$

$$(\Lambda^p\mathfrak{g}^*,\delta^p) \quad \text{and} \quad H^\bullet(\Lambda^p\mathfrak{g}^*,\delta^p)$$

Via a metric g, one gets identified $\mathfrak{g}^* \overset{g}{\cong} \mathfrak{g}$, $\delta^p = \partial^*_{p+1}$, $h_p = \partial_p$

$$\iota\pi - Id_{\Lambda\mathfrak{g}^*} = \partial\partial^* + \partial^*\partial =: \Delta \qquad \ker\Delta = H^\bullet(\mathfrak{g},\mathbb{K})$$

□

Higher products on $H^\bullet(\mathfrak{g}, \mathbb{K})$

Theorem

$H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ *is generated in degree 1 as C_∞-algebra.*

$$H^\bullet(\mathfrak{g}, \mathbb{K}) \cong H^n(\Lambda\mathfrak{g}^*, \delta) \cong \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K})(V^*) \cong \bigoplus_{\lambda:\lambda=\lambda'} S^\lambda(V^*)$$

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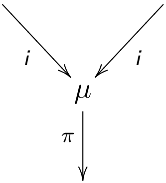
$$m_2(x, y) := \pi\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$



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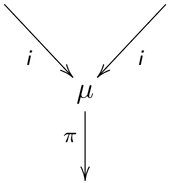
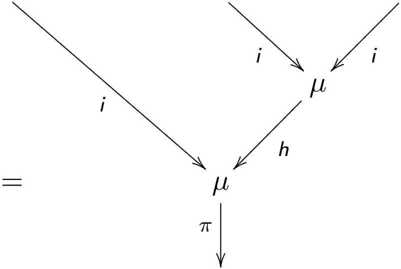
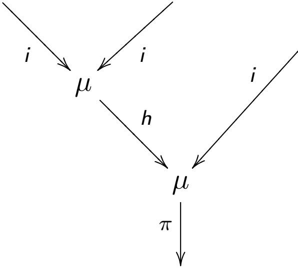
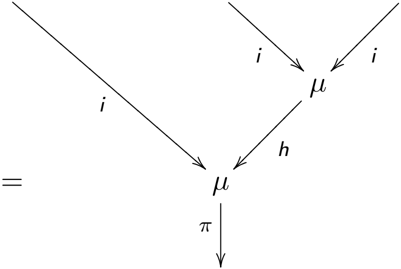


$$m_2(\boxed{\alpha_1}, \boxed{\alpha_2}) = \pi(\boxed{\alpha_1} \wedge \boxed{\alpha_2}) = \pi\left(\begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \end{array}\right) = 0$$

$m_3 =$ 

$-$ 

$m_3 =$



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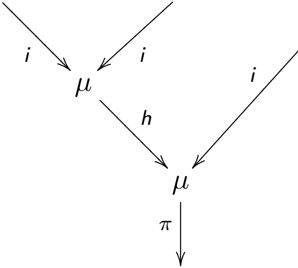

$$m_3\left(\boxed{\alpha_1}, \boxed{\alpha_2}, \boxed{\alpha_3}\right) = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \\ \cline{1-1}\end{array} \in H^2(\mathfrak{g}, \mathbb{K})$$

$$m_3\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \alpha_4 \\ \hline \end{array}, \begin{array}{|c|}\hline \alpha_5 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_4 & \multicolumn{2}{|c}{} \\ \cline{1-1} \alpha_5 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \quad \in H^3(\mathfrak{g}, \mathbb{K})$$

$$m_3\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline \alpha_4 \\ \hline\end{array}, \begin{array}{|c|}\hline \alpha_5 \\ \hline\end{array}\right) = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_4 & \multicolumn{2}{c}{} \\ \cline{1-1} \alpha_5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

$$m_2\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline \alpha_4 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \alpha_4 \\ \hline\end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

Reference

M. Dubois-Violette, T. Popov. Young tableaux and homotopy commutative algebra. arXiv:1202.2230

$$m_3\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \alpha_4 \\ \hline \end{array}, \begin{array}{|c|}\hline \alpha_5 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_4 \\ \cline{1-1} \alpha_5 \\ \cline{1-1} \end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

$$m_2\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \alpha_4 \\ \hline \end{array}\right) = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \alpha_4 \\ \hline \end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

Reference

M. Dubois-Violette, T. Popov. Young tableaux and homotopy commutative algebra. arXiv:1202.2230

Conjecture on higher projects

All products of the C_∞-algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ besides m_2 and m_3 are **trivial**!

$$m_k(\alpha_1, \ldots, \alpha_k) = 0 \qquad k \geq 4$$