West University of Timişoara
Faculty of Physics

Aspects of quantum modes on de Sitter spacetime

speaker: Gabriel Pascu

Belgrade
11 Sept. 2012

Introduction

openstreetmap.org

Universitatea de Vest din Timişoara

West University of Timişoara
Faculty of Physics

Gabriel Pascu

- PhD student
- thesis: "Contributions to the quantum field theory on de Sitter spacetime"
- adviser: prof. dr. Ion I. Cotăescu

Outline of the Short-Talk

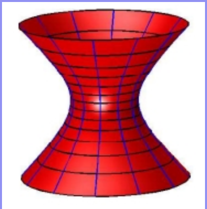
1. de Sitter spacetime
   - The de Sitter background
   - Charts on dS spacetime
2. Quantum Modes
   - de Siter spacetime and modes
   - Conserved operators
   - Defining the quantum modes
   - Modes on the euclidean chart
   - Spherical energy basis modes
3. Concluding remarks

The de Sitter manifold

- can be embedded in a 5D Minkowski space

Constraint:

$$\eta_{AB} Z^A Z^B = -\frac{1}{\omega^2}$$



		embedding	
manifold	$\mathbb{M}^5$		$d\mathbb{S}$
		$\longrightarrow$	
coords.	$\{Z^A\}$		$\{x^\mu\}$
		$Z^A = Z^A(x^\mu)$	
metric	η^{AB}		$g^{\mu\nu}$

Metric tensor and Killing vectors

Induced metric on dS

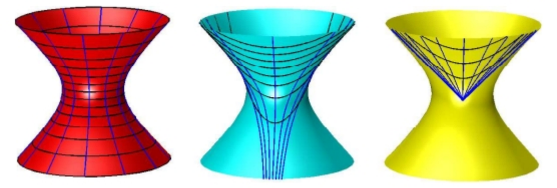
$$g_{\mu\nu} = \eta_{AB} \frac{\partial Z_A}{\partial x^\mu} \frac{\partial Z_B}{\partial x^\nu}$$

- inherits its isometry group from the gauge group of $\mathbb{M}^5$: $SO(1,4)$
- is a maximally symmetric spacetime (has 10 Killing vectors)

$$k^\mu_{AB} = g^{\mu\nu} Z_A \overleftrightarrow{\partial}_\nu Z_B$$

FLRW charts

- dS manifold is the only one that admits all 3 types of FLRW charts
- important for cosmology: exhibit isotropy (rotational symmetry is manifest) and homogeneity
- ∂_t is not a Killing vector

[1]

$k = +1$	$k = 0$	$k = -1$
hyperspherical	spatially flat	hyperbolic

[1] Moschella, Progr.Math.Phys.47:120 (2006)

static chart $\{t_s, r_s, \theta, \phi\}$

$$ds^2 = (1-\omega^2 r_s^2)dt_s^2 - \frac{dr_s^2}{1-\omega^2 r_s^2} - r_s^2 d\Omega_2^2$$

- ∂_{t_s} is a Killing vector

dS- Painlevé chart [2] $\{t, r_s, \theta, \phi\}$

$$ds^2 = (1-\omega^2 r_s^2)dt^2 + 2\omega r_s dr_s dt - dr_s^2 - r_s^2 d\Omega_2^2$$

- hybrid between static chart and FLRW chart
- time slices are euclidean spaces
- ∂_t is a Killing vector

[2] Gautreau, Phys.Rev.D27:764 (1983) / Parikh, Phys.Lett.B546:189 (2002)

'Natural' charts

- time and space on equal footing $Z^\mu = \frac{x^\mu}{f(x)}$
- basis for a so-called 'de Sitter-invariant special relativity':
- Beltrami chart [3]: $f(x) = \sqrt{1-\omega^2(t^2-\vec{x}^2)}$- the "inertial coordinates" of dS
- stereographic chart [4]: $f(x) = 1-\omega^2(t^2-\vec{x}^2)/4$- conformal to Minkowski spacetime
- NO symmetries are manifest

---

[3] Guo, Huang, Xu, Zhou, Mod.Phys.Lett.A19:1701 (2004)

[4] Aldrovandi, Almeida, Pereira, Class.Quantum Grav.24:1385 (2007)

Quantisation of fields on de Sitter spacetime

- gravity remains classical, only matter fields are quantized!
- dS background - arena of interactions for the quantum theory (i.e. fields do NOT interact with the background)

Minkowski spacetime $\to$ deSitter Spacetime

- no. of symmetries remains the same! (10 Killing Vectors) - give rise to conserved operators
- parameter introduced: ω (related to the cosmological constant). In the limit $\omega \to 0$, Minkowski quantities should be obtained
- the complexity of equations and their solutions increases
- first step: determine the free fields on the dS manifold

The conserved operators

Scalar conserved operators:

$$X_{AB} = -ik^{\mu}_{AB}\partial_{\mu}$$

Hamiltonian operator:

$$H = \omega X_{04}$$

Angular momentum:

$$J_i = i\varepsilon_{ijk}X_{jk}$$

Momentum operator:

$$P_i = \omega(X_{i4} + X_{0i})$$

'Runge-Lenz-type' operator:

$$R_i = X_{i4}$$

The 'correct' momentum operator on dS

Minkowski limit:

$$\lim_{\omega\to 0} P_i = \lim_{\omega\to 0} \omega R_i = P_i^{\mathbb{M}} \equiv -i\partial_i$$

Commutation relations:

$$[P_i, P_j] = 0$$

$$[R_i, R_j] = i\varepsilon_{ijk} J_k$$

- P_i are good momentum operators
  Also:

$$[H, P_i] = i\omega P_i$$

- Energy and momentum can't be measured simultaneously on dS

- The field operator can be expanded as $\Phi(x) = \int da\,db\,dc\ f_{abc}(x)a(a,b,c) + f^*_{abc}(x)a^\dagger(a,b,c)$, and must satisfy:

Field equation- s=0 (Klein-Gordon equation)

$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Phi(x)) - m^2\Phi(x) = 0$$

- AND

Eigenvalue equations for operators:

$$A\Phi(x) = a\Phi(x), \qquad B\Phi(x) = b\Phi(x), \qquad C\Phi(x) = c\Phi(x)$$

- The latter give the separating constants a physical meaning: quantities arising from measurements corresponding to a

C.S.C.O. (complete set of commuting operators)

$$\{\mathcal{E}, A, B, C\}$$

Spatially flat FLRW (euclidean) chart

Metric

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x} \cdot d\vec{x}$$

Conserved operators:

- spatial translations are manifest

$$P_i = -i\partial_i$$

- time translation accompanied by a spatial dilatation (in accordance with the concept that dS spacetime is expanding)

$$H = -i(\partial_t + x^i \partial_i)$$

- P_i do not commute with $H \Rightarrow$ two kinds of mode functions

De Sitter spacetime with $ds^2 = dt^2 - e^{2\omega t} d\vec{x} \cdot d\vec{x}$

-momentum-basis plane waves [5]:

$$f_{\vec{p}}(t,\vec{x}) = \frac{1}{2}\sqrt{\frac{\pi}{\omega}}\frac{1}{(2\pi)^3} e^{-\frac{3\omega t}{2}} e^{\frac{i\pi\nu}{2}} H_\nu^{(1)}\left(\frac{p}{\omega}e^{-\omega t}\right) e^{i\vec{p}\cdot\vec{x}}$$

-energy-basis plane waves [6]:

$$f_{E,\vec{n}}(t,\vec{x}) = \frac{1}{2}\sqrt{\frac{\omega}{2}}\frac{1}{(2\pi)^3} e^{-\frac{3\omega t}{2}} e^{\frac{i\pi\nu}{2}} \int_0^\infty \sqrt{s} H_\nu^{(1)}(se^{-\omega t}) e^{i\omega s\vec{n}\cdot\vec{x} - i\frac{E}{\omega}\ln s}$$

[5] Nachtmann, Commun.math.Phys.6:1 (1967)

[6] Cotăescu, Crucean, Pop, Int.J.Mod.Phys.A23:2563 (2008)

The 'Schrödinger Picture' formalism[7]

Apply a (possibly non-unitary) operator $U(x)$

$$\Phi(x) \to \Phi_S(x) = U(x)\Phi(x)$$
$$O \to O_S = U(x) O U(x)^{-1}$$

where

$$U(x) = e^{-\omega t (x^i \partial_i)}$$

such that

$$U(x) F(x^i) U(x)^{-1} = F(e^{-\omega t} x^i)$$
$$U(x) F(\partial_i) U(x)^{-1} = F(e^{\omega t} \partial_i)$$

[7] Cotăescu, arXiv:0708.0734 (2007)

Deriving the equation

Klein-Gordon eq. in $\{t, x, y, z\}$ chart

$$\left(\partial_t^2 + 3\omega\partial_t - e^{-2\omega t}\Delta_{x,y,z} + m^2\right)\Phi(t,\vec{x}) = 0$$

Natural Picture $\to$ Schrödinger Picture:

$$\partial_t \to \partial_t + \omega x^i \partial_i$$

$$\partial_i \to e^{\omega t}\partial_i$$

$$\Delta \to e^{2\omega t}\Delta$$

$$\Phi(t,\vec{x}) \to \Phi_S(t,\vec{x})$$

Klein-Gordon eq. in $\{t, x, y, z\}$ chart - Schrödinger Picture

$$\left((\partial_t + \omega x^i \partial_i)^2 + 3\omega(\partial_t + \omega x^i\partial_i) - \Delta_{x,y,z} + m^2\right)\Phi_S(t,\vec{x}) = 0$$

$\{t,\vec{x}\} \to \{t,r,\theta,\phi\}$:

$$x^i\partial_i = r\partial_r$$

$$\Delta_{x,y,z} = \Delta_{r,\theta,\phi} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\Delta_{\theta,\phi}}{r^2}$$

Klein-Gordon eq. in $\{t,r,\theta,\phi\}$ chart - Schrödinger Picture

$$\left((\partial_t + \omega r\partial_r)^2 + 3\omega(\partial_t + \omega r\partial_r) - \partial_r^2 - \frac{2}{r}\partial_r - \frac{\Delta_{\theta,\phi}}{r^2} + m^2\right)\Phi_S(t,r,\theta,\phi) = 0$$

Solution (back in Natural Picture)

$$f_{E,l,m_l}(t,r,\theta,\phi) = Ne^{-iEt}(\omega r e^{\omega t})^l {}_2F_1(\alpha,\beta;l+3/2;\omega^2 r^2 e^{2\omega t})Y_{l,m_l}(\theta,\phi)$$

Solution- in integral representation (via Hankel transform)[8]

$$f_{E,l,m_l}(t,r,\theta,\phi) = N \times 2^{-i\varepsilon}\frac{i\pi}{2}e^{\frac{i\pi\nu}{2}}\frac{\Gamma(l+3/2)}{\Gamma(\alpha)\Gamma(\beta)}Y_{l,m_l}(\theta,\phi)e^{-\frac{3\omega t}{2}}\frac{1}{\sqrt{\omega r}}\times$$
$$\times\int_0^\infty s^{-i\varepsilon}H_\nu^{(1)}(se^{-\omega t})J_{l+1/2}(\omega rs)ds$$

Scalar Product (in spherical coordinates)

$$\langle f_{E,l,m_l}, f_{E',l',m_l'}\rangle = i\int_0^\infty d^3r\, r^2\int d\Omega e^{3\omega t} f^*_{E,l,m_l}(t,r,\theta,\phi)\overleftrightarrow{\partial_t} f_{E',l',m_l'}(t,r,\theta,\phi)$$

$$N = \sqrt{\frac{\omega}{2}\frac{1}{\pi}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(l+3/2)}$$

[8]Mod.Phys.Lett.A27:1250124 (2012)

Concluding remarks

- the limiting case $\omega \to 0$ can't be evaluated for the k=0 FLRW chart quantum modes, but that's OK: modes are not observable quantities
- on a chart there can be more than one useful mode-expansion of the field operator
- while the energy-basis modes are different from the momentum-basis ones, there is still no Bogolyubov mixing (vacuum is stable under transf. from one set to another)
- the most useful charts for computing quatum modes are the ones where symmetries are manifest
- spatially flat FLRW chart- translational symmetries are manifest $\Phi(x) \sim e^{i\vec{p}\cdot\vec{x}}$ - use of free modes in a QFT with perturbative Feynman-Dyson formalism [9]

[9] Crucean, Mod.Phys.Lett.A22:2573 (2007)
Cotăescu, Crucean, Int.J.Mod.Phys.A23:1351 (2008)

Thank you for your attention!