Space-time models with dust and cosmological constant, that allow integrating the Hamilton-Jacobi test particle equation by separation of variables method.

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Space-times models with dust and radiation

$$T_{ij} = \Lambda g_{ij} + \rho u_i u_j + \varepsilon l_i l_j,$$

$$g^{ij} u_i u_j = 1, \qquad g^{ij} l_i l_j = 0, \qquad i,j = 1,...n$$

signature is $(+,-,-,-)$

Λ – cosmological constant,

ρ – mass density of dust matter,

u_i – velocity of dust matter,

ε – energy density of radiation,

l_i – wave vector of radiation.

Equation of test particle in Hamilton-Jacobi form

$$g^{ij} S_{,i} S_{,j} = m^2 \qquad i,j = 1,...n$$

STÄCKEL SPACES

Definition 1 *Let V_n be a n–dimensional Riemannian space with metric tensor g_{ij}. The Hamilton – Jacobi equation*

$$g^{ij}S_{,i}S_{,j} = m^2 \qquad i,j = 1,...n \tag{1.1}$$

can be integrated by **complete separation of variables method** *if co-ordinate set $\{u^i\}$ exists for which complete integral can be presented in the form:*

$$S = \sum_{i=1}^{n} \phi_i(u^i, \lambda) \tag{1.2}$$

where $\lambda_1...\lambda_n$ – is the essential parameter.

Definition 2 *V_n is called the Stäckel space if the Hamilton–Jacobi equation (1.1) can be integrated by complete separation of variables method.*

Theorem 1 *Let V_n be the Stäckel space. Then g_{ij} in privileged co-ordinate set can be shown in the form*

$$g^{ij} = \sum_{\nu} (\Phi^{-1})^{\nu}_{n} G^{ij}_{\nu},$$

$$G^{ij}_{\nu} = G^{ij}_{\nu}(u^{\nu}), \qquad \Phi^{\nu}_{\mu} = \Phi^{\nu}_{\mu}(u^{\mu}), \qquad (2.1)$$

$$G^{ij}_{\nu} = \delta^{i}_{\nu}\delta^{j}_{\nu}\varepsilon_{\nu}(u^{\nu}) + \sum_{p} (\delta^{i}_{\nu}\delta^{j}_{p} + \delta^{j}_{\nu}\delta^{i}_{p}) G^{\nu p}_{\nu} + \sum_{p,q} \delta^{i}_{p}\delta^{j}_{q} G^{pq}_{\nu},$$

(no summation over ν)

$$i, j = 1, ...n, \quad p, q = 1, ...N, \quad \nu, \mu = N+1, ...n.$$

$$1 \leq N < n$$

where $\Phi^{\nu}_{\mu}(u^{\mu})$ – is called the Stäckel matrix.

Geodesic equations of Stäckel spaces admit **the first integrals** that commutes pairwise with respect to the Poisson bracket

$$\underset{\mu}{X} = (\Phi^{-1})^{\nu}_{\mu}(\varepsilon_{\nu} p^2_{\nu} + 2G^{\nu p}_{\nu} p_p p_{\nu} + h^{pq}_{\nu} p_p p_q),$$

$$\underset{p}{Y} = \underset{p}{Y}{}^{i} p_i, \qquad (2.2)$$

$$p, q = 1, ...N; \quad \nu, \mu = N+1, ..., n.$$

$\Phi^{\nu}_{\mu}(u^{\mu})$ – is called the Stäckel matrix, functions ε_{ν}, $G^{\nu p}_{\nu}$, h^{pq}_{ν} depends only from u^{ν}, p_i – momentum.

If we write the functions X_{ν}, Y_p in the form:

$$\underset{\nu}{X} = \underset{\nu}{X}{}^{ij} p_i p_j, \qquad \underset{p}{Y} = \underset{p}{Y}{}^{i} p_i \qquad (2.3)$$

then

$$\underset{\nu}{X}{}_{(ij;k)} = \underset{p}{Y}{}_{(i;j)} = 0$$

(the semicolon denotes the covariant derivative and the brackets denote symmetrization). Therefore $Y_p{}^i$, $X_{\nu}{}^{ij}$ are the components of **vector and tensor Killing fields** respectively.

Definition 3 *Pairwais commuting Killing vectors* ${Y_p}^i$*, where* $p = 1, ...N$ *and Killing tensors* ${X_\nu}^{ij}$*, where* $\nu = N+1, ...n$ *form a so called complete set of the type* $(N.N_0)$*, where*

$$N_0 = N - rank|| \underset{p}{Y}{}^i \underset{q}{Y}{}_i||$$

Theorem 2 *A necessary and sufficient geometrical criterion of a Stäckel space is the presence of a complete set of the type* $(N.N_0)$*.*

Then the Hamilton - Jacobi equation can be integrated by the complete separation of variables method if and only if the complete set of the first integrals exists.

Definition 4 *Space - time is called the Stäckel one of the type* $(N.N_0)$ *if the complete set of the type* $(N.N_0)$ *exists.*

Let us consider the Hamilton–Jacobi equation for the charged particle

$$g^{ij}(S_{,i}+A_i)(S_{,j}+A_j)=m^2. \tag{2.8}$$

Theorem 3 *If eq.(2.8) admits complete separations of variables then g^{ij} is the metric tensor of the Stäckel space type $(N.N_0)$.*

Using this theorem one can show that the separation takes place for the same privileged coordinate set and

$$A^i=(\Phi^{-1})^{\nu}_{n}h^i_{\nu}(u^{\nu}), \qquad A_iA^i=(\Phi^{-1})^{\nu}_{n}h_{\nu}(u^{\nu}). \tag{2.9}$$

The last condition can be regarded as an additional functional equation.

Stäckel space-time is called the **special** one if (2.8) admits complete separation of variables.

Separation of variables for the Klein – Gordon – Fock equation.

Let us consider the Klein–Gordon–Fock equation for the Riemannian space.

$$(\widehat{\mathcal{H}} - m^2)\psi \equiv [-g^{kl}\nabla_k\nabla_l - m^2]\psi = 0 \qquad (3.1)$$

∇_i - is covariant derivative, $\psi -$ scalar function.

Definition 5 *Eq.(3.1) admits complete separation of variables if co-ordinate set $\{u^i\}$ exists for which the complete solution can be presented in the form*

$$\psi = \prod_{i=1}^{n} \phi_i(u^i, \lambda), \qquad det||\frac{\partial^2\psi}{\partial u^i \partial\lambda_j}|| \neq 0. \quad (3.2)$$

Theorem 4 *Let Klein–Gordon–Fock equation admits complete separation of variables. Then* g^{ij} *is a metric tensor of a Stäckel space. Moreover the separation of variables takes place at the same privileged co - ordinate set.*

It was proved that in the special Stäckel electrovac spacetimes Klein–Gordon–Fock equation can be integrated by the complete separation of variables method.

(Recall that Stäckel space - time is called the **special** one if HJ-equation with A_i admits complete separation of variables)

Stäckel spaces and field equations of the theories of gravity.

The metrics of the Stäckel spaces can be used for integrating the field equations of General Relativity and other metric theories of Gravity.

Note that such famous GR exact solutions as Schwarcshild, Kerr, NUT, Friedman belong to the class of Stäckel spaces.

At the moment all Stäckel spaces satisfying the Einstein–Maxwell equations have been found in our papers (Bagrov, Obukhov, Shapovalov, Osetrin) .

Classification of Stackel space-times for the following theories have been considered in our papers:

1. Brans–Dicke scalar-tensor theory. The field equations have the form

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{\phi}T_{ij} - \frac{\omega}{\phi^2}(\phi_{;i}\phi_{;j} -$$

$$- \frac{1}{2}g_{ij}\phi_{;k}\phi^{;k}) - \frac{1}{\phi}(\phi_{;ij} - g_{ij}\Box\phi) \qquad (5.1)$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T^{i}{}_{i}, \quad \Box = g^{ij}\nabla_i\nabla_j, \quad \omega = const.$$

2. The classification problem for the Einstein–Vaidya equations when the stress–energy tensor have the form:

$$T_{ij} = a(x)\, l_i l_j, \qquad l_i l^i = 0 \qquad (5.2)$$

Conformally Stäckel spaces.

Let us consider the Hamilton–Jacobi equation for a massless particle

$$g^{ij}S_{,i}S_{,j} = 0 \tag{6.1}$$

Obviously this equation admits complete separation of variables for a Stäckel space. Yet one can verify that if g^{ij} has the form

$$g_{ij} = \tilde{g}_{ij}(x)\exp 2\omega(x) \tag{6.2}$$

where $\tilde{g}_{ij}$ is a metric tensor of the Stäckel space, then eq.(6.1) can be solved by complete separation of variables method too.

Note that conformally Stäckel spaces play important role when massless quantum equations are considered (f.e. conformal invariant Chernikov–Penrose equation, Weyl's equation etc.).

The problem of classification of conformally Stäckel spaces satisfying the Einstein equation

$$R_{ij} = \Lambda g_{ij}, \qquad \Lambda = const \tag{6.3}$$

$$g_{ij} = \tilde{g}_{ij}(x) \exp 2\omega(x)$$

is more difficult than appropriate problem for the Stäckel spaces.

We obtained the following form of integrability conditions:

$$\tilde{\nabla}_\delta \left(\tilde{C}^\delta{}_{\alpha\beta\gamma} \exp{(n-3)\omega} \right) = 0. \tag{6.4}$$

If dimension of the space V_n equals to 4, eq. (6.4) has the form

$$\tilde{\nabla}^\delta \left(\tilde{C}_{\delta\alpha\beta\gamma} \exp \omega \right) = 0. \tag{6.5}$$

Using integrability conditions we have proved the following theorem

Theorem 5 *Let g_{ij} be the metric tensor of the Stäckel space of type (N.1). Then Einstein space conformal to $\tilde{V}_4$ admits the same Killing vectors as V_4.*

Moreover one can prove the following statement.

Theorem 6 *Let $\tilde{V}_n$ is conformally Stäckel space of type (N.1) ($N \geq 2$) satisfying the Einstein equation (6.3). Then Hamilton–Jacobi equation (1.1) admits the complete separation of variables.*

In other words nontrivial null conformally Stäckel solutions of the Einstein equations belong only to (1.1)-type Stackel spaces.

HOMOGENEOUS STÄCKEL SPACES

Let us consider the problem of classification of space-homogeneous models of space-times which admit a complete separation of variables in Hamilton-Jacobi equation.

The most interesting models for cosmology are space-homogeneous models, which admit a 3-parametrical transitive group of motions with space-like orbits.

On the other hand, the Stäckel space $(N.N_0)$ type admit N Killing vectors.

Thus, there is a problem of finding a subclass of homogeneous space-times admitting complete sets of integrals of motion.

In other words, a space-time with a complete set must admit a 3-parametrical transitive group of motions with space-like orbits.

Let us consider Stackel space of type (3.1). This type of Stackel space-times is rather interesting in this context, because its metric depend only on such variable of privileged coordinate set, which corresponds to null (wave) hypersurface of the Einstein equation.

In other words, the Stackel space-times of this type are common to spaces filled of radiation (gravitational, electromagnetic etc.).

In a privileged coordinate set metric of (3.1) type has the form

$$\mathrm{g}^{ij} = \begin{pmatrix} 0 & 1 & b_2(x^0) & b_3(x^0) \\ 1 & 0 & 0 & 0 \\ b_2(x^0) & 0 & a_{22}(x^0) & a_{23}(x^0) \\ b_3(x^0) & 0 & a_{23}(x^0) & a_{33}(x^0) \end{pmatrix},$$

where x^0 is the wave-like null variable.

Space admit 3 commuting Killing vectors

X_1, X_2, X_3; $\qquad [X_p, X_q] = 0, \qquad p, q, r = 1, 2, 3$

with components

$$X_p{}^i = \delta_p^i.$$

The metric projection on orbits of this group of motions is degenerated. Thus, we need an additional Killing vector

$$X_4^i = \xi^i.$$

The commutative relations of group X_1 — X_4 have the form

$$\begin{aligned}
&[X_m, X_4] = \alpha_m X_4 + \beta_m{}^n X_n \\
&[X_1, X_4] = \alpha_1 X_4 + \beta_1{}^p X_p, \qquad p, q = 1, 2, 3 \\
&[X_p, X_q] = 0
\end{aligned}$$

The Jacobi identities for the structure constants have the form

$$\begin{aligned}
&\beta_1{}^1 \alpha_m = 0 \\
&\alpha_2 \beta_3{}^n = \alpha_3 \beta_2{}^n \\
&\alpha_m \beta_1{}^n = \alpha_1 \beta_m{}^n
\end{aligned}$$

Classification of obtained solution by Bianchi:

	Stäckel spaces (3.1)						
	1	2	3	4	5	6	7
I	+	+		+			
II						+	
VII_0							+
VI_0	+	+	+				
VIII							
IX							
V	+	+	+		+		
IV					+		
VII_a							+
III	+	+	+	+			
VI_a	+	+	+				

Metrics of Bianchi types VIII and IX are absent.

Example of Application
of obtained metrics for radiation.

Einstein-Vaidya equations have the form:

$$R_{\alpha\beta} = q^2(x)\, l_\alpha l_\beta, \qquad l_\alpha l^\alpha = 0,$$

where $q(x)$ – energy density of radiation,
l_α – wave vector.

This is easy to see that the obtained metrics satisfy field equations at the only condition for the energy density of radiation. For all our metrics we have

$$(ql)^2 = \frac{k_1}{(x^0)^2} + k_2,$$

where k_1 and k_2 are constants. For type A metrics we have $k_2 = 0$ and for type B metrics $k_1 = 0$.

The obtained solution represents an analogue of spherical wave (for $k_2 = 0$) or an analogue of homogeneous radiation (for $k_1 = 0$) for 7 types of homogeneous spaces (and 9 type of cosmological models by Bianchi classification).

Conformally-flat Stackel space-times

As a simple generalization of flat space-time, including the famous cosmological model as Friedmann-Robertson-Walker, De Sitter space-times, we can consider a conformally–flat space.

Then the first step of classification – selection of conformally–flat metrics by condition:

$$C_{ijkl} = 0,$$

where C_{ijkl} - Weyl tensor.

Stackel metric of type (2.0)
(coordinate set, allowing separation of variables)

Stackel metric of type (2.0) in a privileged coordinate set can be reduced to the following convenient form:

$$g^{ij} = \frac{1}{\Delta}\begin{pmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$$

where

$$A = a_2(x^2) + a_3(x^3), \qquad B = b_2(x^2) + b_3(x^3),$$

$$C = c_2(x^2) + c_3(x^3), \quad \epsilon = \pm 1, \quad \Delta = \tau_2(x^2) + \tau_3(x^3)$$

Classes of conformally–flat metrics (2.0) type:

$$g^{ij} = \frac{1}{\Delta}\begin{pmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$$

Various cases of dependence of functions A, B, C, taking into account the coordinate transformation, lead to the existence of four classes of solutions:

1. $C = 0, \quad A \neq 0.$

2. $B = 0.$

3. $A = \lambda C, \quad BC \neq 0, \quad \lambda \neq 0, \quad B \neq \kappa C.$

4. $B = \lambda A + \mu C,\ \lambda\mu > \frac{1}{4}, \quad AC \neq 0, \quad A \neq \kappa C.$

where λ, μ and κ are constants.

Conformally–flat Stackel metrics (2.0) type (coordinate set, allowing separation of variables)

$$g^{ij} = \frac{1}{\Delta}\begin{pmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix},$$

1. The case $C = 0$, $\epsilon = 1$:

$$(1.1) A = \alpha({x^2}^2 + {x^3}^2) + \beta x^2 + \gamma x^3,\ B = 1,$$

$$(1.2) A = \epsilon_1 e^{2\alpha x^2},\ B = e^{\alpha x^2},$$

$$(1.3) A = \epsilon_1({x^2}^4 - {x^3}^4),\ B = {x^2}^2 + {x^3}^2,$$

$$(1.4) A = \epsilon_1(\cos 2x^2 + \text{ch}\ 2x^3),\ B = \sin x^2 + \epsilon_2 \text{ch}\ x^3,$$

$$\epsilon_1, \epsilon_2 = \pm 1$$

Small Greek letters denote a constants.

Conformally–flat Stackel metrics (2.0) type (coordinate set, allowing separation of variables)

$$g^{ij} = \frac{1}{\Delta}\begin{pmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix},$$

2. The case $B = 0$:

(2.1) $\quad A = \dfrac{\beta c_2 + \delta}{\alpha c_2 + \gamma} + \dfrac{\beta c_3 - \delta}{-\alpha c_3 + \gamma}, \quad \epsilon = \pm 1$

where function c_μ satisfies:

$$c_2'^2 = \frac{1}{2}(\alpha c_2 + \gamma)(\kappa c_2{}^2 + \lambda c_2 + \mu),$$

$$c_3'^2 = \frac{1}{2}\epsilon(\alpha c_3 - \gamma)(\kappa c_3{}^2 - \lambda c_3 + \mu).$$

(2.2) $\quad A = 1/x^{2^2},\ C = -\epsilon/x^{3^2}$

(2.3) $\quad A = 1/\sin^2 x^2,\ C = -\epsilon/\sin^2_\epsilon x^3$

$$\sin_\epsilon x = \begin{cases} \sin x, & \epsilon = -1 \\ \mathrm{sh}\ x, & \epsilon = 1 \end{cases}$$

Solutions with dust and cosmological constant (coordinate set, allowing separation of variables)

$$R_{ij} - \frac{1}{2} R g_{ij} = \Lambda g_{ij} + \rho u_i u_j, \qquad g^{ij} u_i u_j = 1$$

1.

$$dS^2 = C \left(\frac{1}{A} {dx^0}^2 + \frac{1}{C} {dx^1}^2 + {dx^2}^2 + {dx^3}^2 \right),$$

where $C = c_2 + c_3$, $A > 0$, $C < 0$,

$$A = \frac{\beta c_2 + \delta}{\alpha c_2 + \gamma} + \frac{\beta c_3 - \delta}{-\alpha c_3 + \gamma},$$

where c_2, c_3 are solution of

$${c_2'}^2 = \frac{1}{2} (\alpha c_2 + \gamma)(\kappa {c_2}^2 + \lambda c_2 + \mu),$$

$${c_3'}^2 = \frac{1}{2} \epsilon (\alpha c_3 - \gamma)(\kappa {c_3}^2 - \lambda c_3 + \mu).$$

Mass density and 4-velosity of dust:

$$u^i = (0, 1, 0, 0), \qquad \rho = 2\Lambda, \qquad \alpha\beta = 8\Lambda.$$

x^1 is a time variable.

Solutions with dust and cosmological constant (coordinate set, allowing separation of variables)

2.

$$dS^2 = a^2 \left(-\frac{\alpha C + \beta}{C} dx^{0\,2} - \frac{1}{C} dx^{1\,2} - dx^{2\,2} \right) + dx^{3\,2}$$

where $a = a(x^3)$, $C = C(x^2)$, $C > 0$ and

$$C'^2 = \gamma C^2 (\alpha C + \beta).$$

Mass density and 4-velosity of dust:

$$u_i = (0, 0, 0, a), \qquad \rho = \frac{\mu}{a^3},$$

$$a'^2 = \frac{\beta\gamma}{4} + \frac{\Lambda a^2}{3} + \frac{\mu}{3a},$$

$$a'' = \frac{\Lambda a}{3} - \frac{\mu}{3a^2}.$$

x^3 is a time variable.

Conclusion

For dust and radiation models we suggest a method for obtaining analytical solutions in any metric theories of gravity based on the use of coordinate systems that admit separation of variables in the Hamilton-Jacobi equation.

This method is demonstrated for a conformally flat models.

We found 7 types of conformally–flat spaces of Stackel type (2.0).

Metric of class $C = 0$ does not admit solutions with a dust in general relativity.

Metric of class $B = 0$ in general relativity admit two types solutions with a dust.