Categorification of Spin Foam Models

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Spin foams

- Problem with the coupling of fermions:

$$S_\psi = \int_M \epsilon^{abcd} e_a \wedge e_b \wedge e_c \, \bar{\psi} \, \gamma_d \, \nabla \psi \, ,$$

  while $B_{ab} = \epsilon_{abcd} e^c \wedge e^d$.

- Problem with the effective action: the classical limit of the effective action is the area-Regge action [Miković and Vojinović, 2011]. It was conjectured in [Miković and Vojinović, 2011] that the non-geometric configurations are exponentially supressed. No proof yet.

- How to introduce tetrads:
  1. AdS/dS BF theory
  2. Poincare gauge theory
  3. 2-groups

2-groups

- Category: objects and maps (1-morphisms)
- 2-Category: objects, maps and maps between maps (2-morphisms)
- Group = Category with one object and invertible 1-morphisms
- 2-Group = 2-Category with one object and invertible 1 and 2-morphisms
- 2-Group = Crossed module of groups: $(G, H, \partial, \triangleright)$. $G =$ 1-morphisms, $G \times_s H =$ 2-morphisms
- Poincare or Euclidean 2 group: $G = Lorentz$ or $G = SO(4)$, $H = \mathbf{R}^4$

2-BF theory

- $(G, H, \partial, \triangleright) \to (\mathfrak{g}, \mathfrak{h}, \partial, \triangleright)$ = differential crossed module
- $A \in \Omega_1(\mathfrak{g}) \to (A, \beta) \in (\Omega_1(\mathfrak{g}), \Omega_2(\mathfrak{h}))$ = 2-connection
- 2-group gauge transformations: $g : M \to G$ and $\eta : M \to \Omega_1(\mathfrak{h})$

$$A \to g(A + d)g^{-1}\,, \quad \beta \to g^{-1} \triangleright \beta$$

$$A \to A + \partial\eta\,, \quad \beta \to \beta + d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta$$

- 2-curvature

$$F = dA + A \wedge A \to (\mathcal{F}, G) = (F - \partial\beta, d\beta + A \wedge^{\triangleright} \beta)$$

BFCG action and GR

- BFCG action [Girelli,Pfeiffer and Popescu; 2008], [Martins and Miković; 2011]

$$S_0 = \int_M \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge G \rangle_{\mathfrak{h}}$$

is invariant under 2-group gauge transformations if

$$g: \quad B \to g^{-1} B g \,, \quad C \to g \triangleright C \,;$$

$$\eta: \quad B \to B - [C, \eta] \,, \quad C \to C \,.$$

- GR as a constrained BFCG theory for the Poincare 2-group [Miković and Vojinović; 2012]

$$S = \int_M B^{ab} \wedge R_{ab} + e^a \wedge \nabla \beta_a - \lambda^{ab} (B_{ab} - \epsilon_{abcd} e^c \wedge e^d) \,,$$

where $R = d\omega + \omega \wedge \omega$ and $\nabla \beta = d\beta + \omega \wedge \beta$.

Path integral for BFCG

- Discretization of

$$Z = \int \mathcal{D}A\,\mathcal{D}\beta\,\mathcal{D}B\,\mathcal{D}C\,e^{iS_0} = \int \mathcal{D}A\,\mathcal{D}\beta\,\delta(\mathcal{F})\,\delta(G)\,.$$

Let $T(M)$ be a triangulation, then

$$Z = \int_{G^{n_1}} \prod_l dg_l \int_{H^{n_2}} \prod dh_f \prod_f \delta(g_f) \prod_p \delta(h_p)\,,$$

where n_1 is the number of edges l in the dual complex $T^*(M)$ and n_2 is the number of faces f in $T^*(M)$.

$$g_f = \prod_{l\in\partial f} g_l\,, \quad h_p = \prod_{f\in\partial p} \tilde{h}_f\,,$$

where $\tilde{h}_f = h_f$ or $\tilde{h}_f = g_l \triangleright h_f$, p is a polyhedron and $l \in p$.

- In the Poncare/Euclidean 2-group case $h_f = \vec{x}_f \in \mathbf{R}^4$ and

$$\vec{x}_p = \vec{x}_1 + \cdots + g_l \vec{x}_f + \cdots + \vec{x}_n\,.$$

Path integral for BFCG

- By using

$$\delta(g_f) = \sum_{\Lambda_f} dim\,\Lambda_f\,\chi^{(\Lambda_f)}(g_f)\,, \quad \delta(\vec{x}_p) = \frac{1}{(2\pi)^4}\int d^4\vec{L}_p\,e^{i\vec{L}_p\,\vec{x}_p}\,,$$

  one obtains

$$Z = \sum_{\Lambda}\int\prod_p d^4\vec{L}_p\,\prod_l dg_l\,\prod_f dim\,\Lambda_f\,\chi^{(\Lambda_f)}(g_f)$$

$$\prod_f \delta(g_{l(f)}\vec{L}_{p(f)} + g_{l'(f)}\vec{L}_{p'(f)} + g_{l''(f)}\vec{L}_{p''(f)})\,.$$

- $g_1\vec{L}_1 + g_2\vec{L}_2 + g_3\vec{L}_3 = 0$ implies $|\vec{L}_p| = L_p = L_\epsilon$ satisfy the triangle inequalities $\Rightarrow L_\epsilon$ can be identified as the length of an edge ϵ.

State sum for BFCG

- The above state sum/integral is divergent, but it suggests to look for

$$Z = \int_{L\in\mathbf{R}_+^{N_1}} \prod_\epsilon \mu(L_\epsilon)dL_\epsilon \sum_{\Lambda'\in(Irrep(G'))^{N_2}} \sum_{\iota\in(Inttw(\Lambda'))^{N_3}} W(L,\Lambda',\iota)\,,$$

where $L_\epsilon, \Lambda'_\Delta$ and ι_τ are labels for a Poincare/Euclidean 2-group representation, intertwiner and 2-intertwiner, respectively.

- In [Crane and Sheppeard; 2003] and [Baez,Baratin,Freidel and Wise; 2008] it was shown that there are irreps of Poincare/Euclidean 2-group labelled by $L_\epsilon \geq 0$. The corresponding intertwiners are the irreps of $SO(2)$ if L_ϵ form a triangle, and the 2-intertwiners ι_τ are trivial. Hence

$$Z = \int_{L\in\mathbf{R}_+^{N_1}} \prod_\epsilon \mu(L_\epsilon)dL_\epsilon \sum_{m\in\mathbf{Z}^{N_2}} W(L,m)\,.$$

State sum for BFCG

- The results of [Baratin and Friedel; 2007] suggest that

$$W(L,m) = \prod_{\Delta} A_{\Delta}(L) \prod_{\sigma} \frac{\cos S_{\sigma}(L,m)}{V_{\sigma}(L)}\,,$$

and $\mu(L) = L$. Here A_Δ is the area of a triangle Δ , V_σ is the volume of a 4-simplex σ and

$$S_\sigma = \sum_{\Delta \in \sigma} m_\Delta \theta_\Delta(L)\,,$$

where θ_Δ is the interior dihedral angle for σ.

State sum for quantum GR

- Since GR can be considered as a constrained BFCG theory, one can try to impose a discretized analog of $B = (e \wedge e)^*$ constraint. A natural candidate is

$$m_\Delta \, l_P^2 = A_\Delta(L) \, ,$$

since $S_\sigma(m, L)$ becomes the Regge action for σ.

- The results of [Miković and Vojinović; 2011] on the effective action suggest that a good candidate is

$$Z_{GR} = \int_{L \in \mathbf{R}_+^{N_1}} \prod_\epsilon \mu(L_\epsilon) dL_\epsilon \sum_{m \in \mathbf{Z}^{N_2}} \prod_\Delta \delta(m_\Delta l_P^2 - A_\Delta(L)) \prod_\sigma e^{iS_\sigma(m,L)} ,$$

where $\mu(L) \approx L^{-p}$ for large L and $p > 0$.

Conclussions

- Z_{GR} will be a Regge state sum.
- One can obtain a discrete or a continious-length Regge model, depending on how the GR constraint is imposed.
- Effective action and semi-classical limit of Z_{GR}
- Amplitude for matter coupling: $W_{matter} \propto e^{iS_{Regge}(\phi,L)}$.
- Canonical quantization of 2-Poincare GR action
- Categorification of LQG
- Construction of 4-manifold invariants