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Invariant solutions for equations of axion electrodynamics

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Outline

The group classification of possible models of axion electrodynamics with arbitrary self interaction of axionic field is presented. We prove that extension of the basic Poincaré invariance appears for the exponential, constant and trivial interaction terms only. In addition, we use symmetries of axion electrodynamics to find exact solutions for its equations invariant with respect to three parameter subgroups of Poincaré group. As a result we obtain an extended class of exact solutions depending on arbitrary parameters and on arbitrary functions as well. We indicate and discuss possible solutions whose group velocity is higher than the velocity of light. However, their energy velocity are subluminal and so there is not a causality violation.

Introduction

The group analysis of PDEs is a fundamental field including many interesting internal problems. But maybe the most attractive feature of the group analysis is its great value for various applications such as defining of maximal Lie symmetries of complicated physical models, construction of models with a priory requested symmetries, etc. Sometimes the group analysis is the only way to find exact solutions for nonlinear problems.

Introduction

I am going to present you some results obtained with application of the Lie theory to the complicated physical model called axion electrodynamics. Let me start with physical motivations of this research.
To explain the absence of the CP symmetry violation in interquark interactions Peccei and Quinn (Phys. Rev. Lett. **38**, 1440 (1977)) suggested that a new symmetry must be presented. The breakdown of this gives rise to the axion field proposed ten years later by Weinberg (Phys. Rev. Lett. **40**, 223 (1978)) and Wilczek (Phys. Rev. Lett. **40**, 279 (1978)). And it was Wilczek who presented the first analysis of possible physical effects caused by axions in electrodynamics (Phys. Rev. Lett. **58**, 1799 (1987)).

Introduction

Axions belong to the main candidates to form the dark matter. New important arguments to study axionic theories were created in solid states physics. Namely, it was found recently (X-L. Qi, T. L. Hughes, and S-C. Zhang, Phys. Rev. B **78**, 195424 (2008)) that the axionic-type interaction terms appears in the theoretical description of a class of crystalline solids called topological insulators. In other words, although their existence is still not confirmed experimentally axioins are requested at least in three fundamental fields: QCD, cosmology and condensed matter physics. And we decide "to help physicists": make group analysis of axionic theories and find in some sense completed set of the related exact solutions.

Field equations of axion electrodynamics

Let us start with the following model Lagrangian:

$$L = \frac{1}{2}p_\mu p^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu} - V(\theta). \tag{1}$$

Here $F_{\mu\nu}$ is the strength tensor of electromagnetic field, $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $p_\mu = \partial_\mu\theta$, θ is the pseudoscalar axion field, $V(\theta)$ is a function of θ, κ is a dimensionless constant, and the summation is imposed over the repeating indices over the values 0, 1, 2, 3. Moreover, the strength tensor can be expressed via four-potential $A = (A^0, A^1, A^2, A^3)$ as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2}$$

Setting in (1) $\theta = 0$ we obtain the Lagrangian for Maxwell field. Moreover, if θ is a constant then (1) coincides with the Maxwell Lagrangian up to constant and four-divergence terms. Finally, the choice $V(\theta) = \frac{1}{2}m^2\theta^2$ reduces L to the standard Lagrangian of axion electrodynamics.

Let us write the Euler-Lagrange equations corresponding to Lagrangian (1):

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \kappa\mathbf{p}\cdot\mathbf{B},\\ \partial_0\mathbf{E}-\nabla\times\mathbf{B} &= \kappa(p_0\mathbf{B}+\mathbf{p}\times\mathbf{E}),\\ \nabla\cdot\mathbf{B} &= 0,\\ \partial_0\mathbf{B}+\nabla\times\mathbf{E} &= 0, \end{aligned} \tag{3}$$

$$\Box\theta = -\kappa\mathbf{E}\cdot\mathbf{B}+F, \tag{4}$$

where

$$\mathbf{B}=\{B^1,B^2,B^3\},\quad \mathbf{E}=\{E^1,E^2,E^3\},\quad E^a=F^{0a},\quad B^a=\frac{1}{2}\varepsilon^{abc}F_{bc},$$
$$F=\frac{\partial\varphi}{\partial\theta},\quad \Box=\partial_0^2-\nabla^2,\quad \partial_i=\frac{\partial}{\partial x_i},\ \ p_0=\frac{\partial\theta}{\partial x_0},\ \mathbf{p}=\nabla\theta.$$

We make:

- Group classification of equations of motion, where function F is treated as an arbitrary element;
- Construction of an entire family of exact solutions;

In addition we discuss obtained solutions whose group velocity is larger than the velocity of light and prove that they do not lead to causality violation.

Equation (4) includes the free element $F(\theta)$ so we can expect that symmetries of system (3), (4) will depend on explicit form of F. Consider the infinitesimal operator

$$Q = \xi^{\mu}\partial_{\mu} + \eta^{j}\partial_{B^j} + \zeta^{j}\partial_{E^j} + \sigma\partial_{\theta}, \tag{5}$$

and its second prolongation

$$Q_{(2)} = Q + \eta_i{}^{j}\frac{\partial}{\partial B_i^j} + \zeta_i{}^{j}\frac{\partial}{\partial E_i^j} + \sigma_i\partial_{\theta_i} + \eta_{ik}{}^{j}\frac{\partial}{\partial B_{ik}^j} + \zeta_{ik}{}^{j}\frac{\partial}{\partial E_{ik}^j} + \sigma_{ik}\partial_{\theta_{ik}}. \tag{6}$$

Using the infinitesimal invariance criterium we obtain the following determining equations:

$$\begin{gathered}\xi^{\mu}_{B^a} = 0, \quad \xi^{\mu}_{E^a} = 0, \quad \xi^{\mu}_{\theta} = 0,\\ \xi^{\mu}_{x^\mu} = \xi^{\nu}_{x^\nu}, \quad \xi^{\mu}_{x^\nu} + \xi^{\nu}_{x^\mu} = 0, \quad \mu \neq \nu,\end{gathered} \tag{7}$$

$$\begin{gathered}\sigma_{E^a} = 0, \quad \sigma_{B^a} = 0,\\ \sigma_{\theta\theta} = 0,\end{gathered} \tag{8}$$

$$\begin{gathered}\Box\sigma + \left(\sigma_\theta - 2\xi^0_{x^0}\right)\left(F(\theta) + kE^aB^a\right)\\ -k(B^a\zeta^a + E^a\eta^a) - \sigma\dot{F}(\theta) = 0.\end{gathered}$$

$$\Box\xi^\mu - 2\sigma_{\theta x^\mu} = 0, \tag{9}$$

$$
\begin{aligned}
&\xi^a_{x^b} + \eta^b_{B^a} = 0, \quad \xi^a_{x^b} + \zeta^b_{E^a} = 0,\\
&\xi^a_{x^b} - \eta^a_{B^b} = 0, \quad \xi^a_{x^b} - \zeta^a_{E^b} = 0, \quad a \neq b,\\
&\xi^a_{x^0} - \varepsilon_{abc}\eta^c_{E^b} = 0, \quad \xi^a_{x^0} - \varepsilon_{abc}\zeta^c_{B^b} = 0,\\
&\partial_a\eta^a = 0, \quad \partial_a\zeta^a + B^a\partial_a\sigma = 0,\\
&\eta^a_{x^0} + \varepsilon_{abc}\zeta^c_{x^b} = 0, \quad \zeta^a_{x^0} + B^a\sigma_{x^0} - \varepsilon_{abc}(\eta^c_{x^b} + E^b\sigma_{x^c}) = 0,\\
&\eta^a_{x^0} + \varepsilon_{abc}\zeta^c_{x^b} = 0, \quad \zeta^a_{x^0} + B^a\sigma_{x^0} - \varepsilon_{abc}(\eta^c_{x^b} + E^b\sigma_{x^c}) = 0,\\
&\eta^a + B^a\sigma_\theta + \zeta^a_\theta - B^b\zeta^a_{E^b} + \varepsilon_{abc}E^b\xi^0_{x^c} = 0,\\
&\eta^a_{x^0} + \varepsilon_{abc}\zeta^c_{x^b} = 0, \quad \zeta^a_{x^0} + B^a\sigma_{x^0} - \varepsilon_{abc}(\eta^c_{x^b} + E^b\sigma_{x^c}) = 0,\\
&\eta^a + B^a\sigma_\theta + \zeta^a_\theta - B^b\zeta^a_{E^b} + \varepsilon_{abc}E^b\xi^0_{x^c} = 0,\\
&\zeta^a - \eta^a_\theta + E^a\sigma_\theta - E^b\zeta^a_{E^b} - \varepsilon_{abc}B^b\xi^c_{x^0} = 0,\\
&\eta^a_{B^a} - \eta^b_{B^b} = 0, \quad \eta^a_{B^a} - \zeta^b_{E^b} = 0, \quad \eta^a_\theta - B^a\eta^b_{E^b} = 0, \quad \zeta^a_\theta - E^a\eta^b_{E^b} = 0.
\end{aligned}
\tag{10}
$$

Integrating this system we find that for arbitrary F generator Q should be a linear combination of the following operators:

$$P_0 = \partial_0, \quad P_a = \partial_a,$$
$$J_{ab} = x_a\partial_b - x_b\partial_a + B^a\partial_{B^b} - B^b\partial_{B^a} + E^a\partial_{E^b} - E^b\partial_{E^a}, \quad (11)$$
$$J_{0a} = x_0\partial_a + x_a\partial_0 + \varepsilon_{abc}\left(E^b\partial_{B^c} - B^b\partial_{E^c}\right)$$

where ε_{abc} is the unit antisymmetric tensor, $a, b, c = 1, 2, 3$. Operators (11) form a basis of the Lie algebra p(1,3) of the Poincaré group P(1,3). Thus the group P(1,3) is the maximal continuous invariance group of system (3), (4) with the *arbitrary* function $F(\theta)$.

This symmetry can be extended provided function F has one of the following particular forms:

$$F = 0, \;\; F = c \;\texttt{or}\; F = b\exp(a\theta)$$

where $c, \; a$ and b are non-zero constants. The corresponding additional elements of the invariance algebra are:

$$\begin{array}{ll} P_4 = \partial_\theta & \text{if} \quad F(\theta) = c, \\ X = aD - P_4 & \text{if} \quad F(\theta) = be^{a\theta}, \\ P_4 = \partial_\theta, \quad D = x_0\partial_0 + x_i\partial_i - B^i\partial_{B^i} - E^i\partial_{E^i} & \text{if} \quad F(\theta) = 0. \end{array} \tag{12}$$

Operator P_4 generates shifts of dependent variable θ, D is the dilatation operator generating a consistent scaling of dependent and independent variables, and X generates the simultaneous shift and scaling.

Algorithm for finding group solutions

The algorithm for construction of group solutions of partial differential equations goes back to Sophus Lie. Being applied to system (3), (4) it includes the following steps (compare, e.g., with [Olver, 1986]):

- To find a basis of the maximal Lie algebra A_m corresponding to continuous local symmetries of the equation.
- To find the optimal system of subalgebras SA_μ of algebra A_m. In the case of PDE with four independent variables like system (3), (4) it is reasonable to restrict ourselves to three-dimensional subalgebras. Their basis elements have the unified form $Q_i = \xi_i^\mu \partial_\mu + \varphi_i^k \partial_{u_k}, \; i = 1, 2, 3$ where u_k are dependent variables (in our case we can chose $u_a = E_a$, $u_{3+a} = B_a$, $u_7 = \theta$, $a = 1, 2, 3$).

Algorithm for finding group solutions

- Any three-dimensional subalgebra SA_μ whose basis elements satisfy the conditions

$$\mathtt{rank}\{\xi_i^\mu\} = \mathtt{rank}\{\xi_i^\mu, \varphi_i^k\} \tag{13}$$

and

$$\mathtt{rank}\{\xi_i^\mu\} = 3 \tag{14}$$

gives rise to change of variables which reduces system (3), (4) to a system of ordinary differential equations (ODEs). The new variables include all invariants of three parameter Lie groups corresponding to the optimal subalgebras SA_μ.

- Solving if possible the obtained ODEs one can generate an exact (particular) solution of the initial PDEs.
- Applying to this solution the general symmetry group transformation it is possible to generate a family of exact solutions depending on additional arbitrary (transformation) parameters.

Exact solutions

To generate exact solutions of system (3), (4) we can exploit its invariance w.r.t. the Poincaré group. The subalgebras of algebra p(1,3) defined up to the group of internal automorphism has been found for the first by Belorussian mathematician Bel'ko (I.V. Bel'ko, Izv. Akad. Nauk Bel. SSR **1**, 5 (1971)). We use a more advanced classification of these subalgebras proposed by Patera, Winternitz and Zassenhaus (1975) who had specified 30 three-dimensional subalgebras.

Notice that some of these subalgebras do not satisfy conditions (13), (14), and to construct the related exact solutions we develop a special technique which generalizes the weak transversality approach proposed by Grundland, Tempesta, and Winternitz (2003).

Exact solutions

Types of reductions:

- Reductions to algebraic equations
- Reductions to linear ODEs
- Reductions to nonlinear ODEs
- Reductions to PDEs

Examples of exact solutions

Among the constructed exact solutions are plane wave solutions:

$$B_1 = \frac{1}{\sqrt{\nu^2 - 1}}(e_1\theta + b_1), \quad B_2 = \frac{1}{\sqrt{\nu^2 - 1}}(e_2\theta + b_2), \quad B_3 = e_3,$$
$$E_1 = \nu B_2 + \sqrt{\nu^2 - 1}e_1, \quad E_2 = -\nu B_1 + \sqrt{\nu^2 - 1}e_2, \quad E_3 = e_3\theta + b_3$$
$$\theta = a_\nu e^{\nu\omega} + b_\nu e^{-\nu\omega} + \frac{c}{\nu^2}, \quad \omega = x_3 - \nu x_0, \quad \nu > 1$$

where a_ν and b_ν are arbitrary constants.
Solitary wave solution:

$$\theta = \frac{3a}{4\mu}\tanh^2\left(\frac{1}{2}\sqrt{\frac{a}{2}}(x_3 - \nu x_0)\right) \quad \texttt{for} \quad F = \mu\theta^2. \tag{15}$$

Examples of exact solutions

Solutions in radial variables:

$$B_a = \frac{q x_a}{r^3}, \quad E_a = \frac{q\theta x_a}{r^3},$$
$$\theta = \frac{1}{r}\left(C_1 e^{q x_0 + \frac{m}{r}} + C_2 e^{q x_0 - \frac{m}{r}} + C_3 e^{-q x_0 + \frac{m}{r}} + C_4 e^{-q x_0 - \frac{m}{r}}\right). \tag{16}$$

Unusual planar solutions:

$$E_1 = -B_2 = \frac{x_1}{x^3},\ E_3 = 0,\ B_1 = E_2 = \frac{x_2}{x^3},\ B_3 = b,\ \theta = \arctan\left(\frac{x_2}{x_1}\right),$$

where $x^2 = x_1^2 + x_2^2$, b is a constant.

Examples of exact solutions

A particularity of the latest solutions is that they are planar ones. Nevertheless the electric field decreases with growing of x as the field of point charge in the three dimensional space.

Conservation laws

An immediate consequence of symmetries presented above is the existence of conservation laws. Indeed, the system (3), (4) admits a Lagrangian formulation. Thus, in accordance with the Noether theorem, symmetries of equations (3), (4) which keep the shape of Lagrangian (1) up to four divergence terms should generate conservation laws. Let me present the conserved energy-momentum tensor:

$$
\begin{aligned}
T^{00} &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 + p_0^2 + \mathbf{p}^2) + V(\theta), \\
T^{0a} &= T^{a0} = \varepsilon_{abc} E_b B_c + p^0 p^a, \\
T^{ab} &= -E^a E^b - B^a B^b + p^a p^b \\
&+ \frac{1}{2}\delta^{ab}(\mathbf{E}^2 + \mathbf{B}^2 + p_0^2 - \mathbf{p}^2 - 2V(\theta)).
\end{aligned}
\tag{17}
$$

The tensor $T^{\mu\nu}$ is symmetric and satisfies the continuity equation $\partial_\nu T^{\mu\nu} = 0$. Its components T^{00} and T^{0a} are associated with the energy and momentum densities.

It is important to note that the energy momentum tensor does not depend on parameter κ and so is not affected by the term $\frac{\kappa}{4}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu}$ presented in Lagrangian (1). In fact this tensor is nothing but a sum of energy momenta tensors for the free electromagnetic field and scalar field. Moreover, the interaction of these fields between themselves is not represented in (17).

Let us consider one of the obtained plane wave solutions:

$$
\begin{aligned}
&B_1 = c_1 k\theta, \quad B_2 = -c_1\varepsilon, \quad B_3 = 0, \\
&E_1 = -c_1 k, \quad E_2 = -c_1\varepsilon\theta, \quad E_3 = 0, \\
&\theta = a_\mu \cos(\mu(\varepsilon x_0 - k x_3)).
\end{aligned} \tag{18}
$$

Here $\varepsilon,\ k$, and a_μ are arbitrary parameters which should satisfy the following dispersion relations:

$$
(\varepsilon^2 - k^2)(\mu^2 - c_1^2) = m^2. \tag{19}
$$

Let $\mu^2 > c_1^2$ then $(\varepsilon^2 - k^2) = \frac{m^2}{\mu^2 - c_1^2} > 0$. The corresponding group velocity V_g is equal to the derivation of ε w.r.t. k, i.e.,

$$
V_g = \frac{\partial\varepsilon}{\partial k} = \frac{k}{\varepsilon}. \tag{20}
$$

Since $\varepsilon > k$, the group velocity appears to be less than the velocity of light (remember that we use the Heaviside units in which the velocity of light is equal to 1).

On the other hand the phase velocity $V_p = \frac{\varepsilon}{k}$ is larger than the velocity of light, but this situation is rather typical in relativistic field theories.

In the case $\mu^2 < c_1^2$ the wave number k is larger than ε. As a result the group velocity (20) exceeds the velocity of light, and we have a phenomenon of superluminal motion. To understand wether the considered solutions are causal let us calculate the energy velocity which is equal to the momentum density divided by the energy density:

$$V_e = \frac{T^{03}}{T^{00}}. \tag{21}$$

Substituting (18) into (17) we find the following expressions for T^{00} and T^{03}:

$$T^{00} = \frac{1}{2}(\varepsilon^2 + k^2)\Phi + \frac{1}{2}m^2\theta^2, \quad T^{03} = \varepsilon k \Phi$$

where $\Phi = c_1^2(\theta^2 + 1) + \mu^2(a_\mu^2 - \theta^2)$. Thus

$$V_e = \frac{2\varepsilon k \Phi}{(\varepsilon^2 + k^2)\Phi + \frac{1}{2}m^2\theta^2} < \frac{2\varepsilon k}{\varepsilon^2 + k^2} < 1,$$

and this relation is valid for $\varepsilon > k$ and for $\varepsilon < k$ as well.

We see that the energy velocity is less than the velocity of light. Thus solutions (18) can be treated as causal in spite of the fact that for $\mu^2 < c_1^2$ the group velocity is superluminal.

Conclusions

1. We make group classification of equations of axion electrodynamics given by relations (3) and (4).
2. Exact solutions corresponding to three-dimensional subalgebras of the Poincaré algebra has been found. There are 32 types of such solutions defined up to arbitrary constants or arbitrary functions. Some of these solutions can have interesting applications, e.g. for construction of exactly solvable problems for Dirac fermions.
3. Solutions describing the faster-then-light propagation are admissible. However, these solutions are causal since the corresponding energy velocity is subluminal.

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