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Hamiltonian analysis of non-linear massive gravity

7th MATHEMATICAL PHYSICS MEETING:
Summer School and Conference on Modern Mathematical Physics
Belgrade, September 2012

Modifying the gravity?

- The Einstein theory of gravity is extremely successful
- But: 1. Dark Matter is somewhat intriguing

  2. Dark Energy points at a technically very unnatural value of Λ
- Hence, people are looking for IR modifications of GR
- One of the most straight-forward IR modifications amounts to introducing a mass term
- However it is not that easy in GR (ghosts etc)
- !!! Only very recently a potentially healthy modification has been obtained: de Rham – Gabadadze – Tolley non-linear massive gravity

Linearised GR

- At the quadratic level around Minkowski, the EH:

$$-\frac{1}{4}\Big((\partial_\alpha h_{\mu\nu})(\partial^\alpha h^{\mu\nu})-2(\partial^\alpha h_{\mu\nu})(\partial^\nu h^\mu_\alpha)+2(\partial_\alpha h^{\alpha\mu})(\partial_\mu h^\beta_\beta)-(\partial_\mu h^\alpha_\alpha)(\partial^\mu h^\beta_\beta)\Big)$$

- In the usual variables $h_{00}=2\phi$, $h_{0i}=\partial_i b+s_i$ where $\partial_i s_i\equiv 0$, and $h_{ik}=2\psi\,\delta_{ik}+2\partial^2_{ik}\sigma+\partial_i v_k+\partial_k v_i+h^{(TT)}_{ik}$ where $\partial_i v_i\equiv 0$, $\partial_i h^{(TT)}_{ik}\equiv 0$ and $h^{(TT)}_{ii}\equiv 0$ we get up to surface terms

$$-\frac{1}{4}(\partial_\alpha h^{(TT)}_{ik})(\partial^\alpha h^{(TT)}_{ik})+\frac{1}{2}\Big(\partial_k\big(\dot{v}_i-s_i\big)\Big)^2-3\dot{\psi}^2+(\partial_i\psi)^2+4\psi\,\Delta\big(\phi-\dot{b}+\ddot{\sigma}\big)$$

We see the wrong sign kinetic term for ψ, however everything apart from $h^{(TT)}$ is unphysical due to the gauge symmetry. But once we break it, we are to expect some problems to come about!

On the physical variables

- The invariant (physical) quantities are two tensor modes $h_{ik}^{(TT)}$, two vector modes $\dot{v}_i - s_i$, and two scalars, ψ and $\phi - \dot{b} + \ddot{\sigma}$.
- Compare to the usual gauge invariant variables in the cosmological perturbation theory.
- Around the FRW spacetime $ds^2 = a^2(t)\left(dt^2 - d\vec{x}^2\right)$ with conformal time and the “Hubble constant” $H \equiv \frac{\dot{a}}{a}$

$$h_{ik}^{(TT)} \qquad V_i \equiv \dot{v}_i - s_i$$

$$\Phi \equiv \phi - \dot{b} + \ddot{\sigma} - H(b - \dot{\sigma}) \qquad \Psi \equiv \psi + H(b - \dot{\sigma})$$

Fierz-Pauli action

- Generically, a theory of massive gravity has 6 degrees of freedom: 5 of massive spin 2, and 1 of a scalar ghost
- Linearly, there is the Fierz-Pauli mass term with only the first 5 of them:

$$\frac{m^2}{4}\left(h_{\mu\nu}h^{\mu\nu}-h^{\mu}_{\mu}h^{\nu}_{\nu}\right)$$

- There is a well-known problem of vDVZ-discontinuity
- It might very well be cured by Vainstein mechanism
- But, anyway, non-linearly (or, equivalently, around a curved background) it develops the Boulware-Deser ghost (the sixth degree of freedom), be there any Vainstein mechanism or not.
- By performing a clever work with decoupling limits, dRGT have found their non-linear ghost-free action.

Bimetric theories

- Note that in massive gravity theory we introduce an auxiliary metric which is used to raise and lower the indices in the mass term. Otherwise, it would be impossible to construct a non-trivial non-derivative scalar invariant
- This extra metric can be made dynamical by introducing its own Einstein-Hilbert term, and thus we get the bimetric theories with one massless spin two and one massive spin 2 generically accompanied by the ghost
- If $g_{\mu\nu}$ and $f_{\mu\nu}$ are the two metrics, then the basic building block of the potential term is $g^{\mu\alpha} f_{\alpha\nu}$
- Obviously, the same problems persist

Other bimetric theories

- Recently, the actions of the form $\int \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\hat{g})$ have been proposed (Amendola, Enqvist, Koivisto; 1010.4776)
- Unfortunately, the ghosts appear. For fluctuations of g (denote them by h) and the difference between the two metrics (denote it by f) we have at the quadratic level EH(h) – EH(f). (Beltran Jimenez, Golovnev, Koivisto, Karciauskas; 1201.4018) Not very surprising!
- At non-linear level, there are also dynamical vectors and scalars. The conformal mode of the metric difference was shown to be healthy (Koivisto; 1103.2743)
- The tensor ghosts might be cured by combining the new term with the standard GR (1201.4018), but then it flips the sign of conformal mode too.

The ghosts are ubiquitous!

dRGT-gravity

- It is an amusing fact that a family of completely BD ghost free massive gravity models does exist

  (Claudia de Rham, Gregory Gabadadze, Andrew Tolley)

- dRGT have worked with the matrix H defined by

$$g^{-1}f \equiv I - H$$

  and used the Stueckelberg trick in perturbation theory

- The potential can be put into the form $V = 2\mathrm{m}^2\left(\left(\sqrt{g^{-1}f}\right)^{\mu}_{\mu} - 3\right)$
- It is just the first symmetric polynomial (trace) of the eigenvalues of the basic matrix $\sqrt{g^{-1}f}$
- Non-perturbatively, it works if the matrix square root exists and is real. A sufficient condition is that $g^{-1}f$ is positive definite. (C. Deffayet at al.)

Immediate generalizations

- The other possible mass terms are the second and the third order symmetric polynomials of the eigenvalues (the fourth one is just the determinant)

$$V_1 = Tr\sqrt{\cdot}$$

$$V_2 = \left(Tr\sqrt{\cdot}\right)^2 - Tr\left(\sqrt{\cdot}\right)^2$$

$$V_3 = \left(Tr\sqrt{\cdot}\right)^3 - 3\left(Tr\sqrt{\cdot}\right)\cdot\left(Tr\left(\sqrt{\cdot}\right)^2\right) + 2\,Tr\left(\sqrt{\cdot}\right)^3$$

- Generalisation to bimetric version is fairly straightforward since the symmetric polynomials of $\sqrt{g^{-1}f}$ can be expressed in terms of those for $\sqrt{f^{-1}g}$

The ADM formalism

- The standard way to perform a non-perturbative analysis in gravity is to invoke the ADM decomposition

$$ds^2 \equiv -\left(N^2 - N_k N^k\right) dt^2 + 2 N_i dx^i dt + \gamma_{ij} dx^i dx^j$$

- In standard GR the lapse and shift are Lagrange multipliers, and the associated constraints decrease the number of degrees of freedom in γ from 6 to 2.
- In massive gravity the Hamiltonian is non-linear in them, thus leaving generically the full set of 6 independent variables

$$H = -\int d^3 x \sqrt{\gamma} \left(N \left(R^{(3)} + \frac{1}{\gamma} \left(\frac{1}{2} \left(\pi_j^j \right)^2 - \pi_{ik} \pi^{ik} \right) - V \right) + 2 N_i \nabla_k^{(3)} \pi^{ik} \right)$$

where the indices are handled with respect to the metric γ , and π^{ik} are canonical momenta of γ_{ik}

Hassan-Rosen analysis 1

- In dRGT gravity one has to work with the square root of

$$g^{\mu\alpha}\eta_{\alpha\nu} = \begin{pmatrix} \frac{1}{N^2} & \frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}$$

  where we have assumed the Minkowski metric $f = \eta$
- The approach of Hassan and Rosen is to take the root
- It would be fairly simple if it was not for the γ since

$$\begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}^2 = \left(1 - a^k a^k\right) \begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}$$

- How to proceed with the actual case at hand?

Hassan-Rosen analysis 2

- The main idea is to make a clever redefinition of shits

$$N_i = \left(\delta_i^j + N D_i^j(n,\gamma)\right) n_j$$

such that

$$\sqrt{g^{-1}\eta} = \frac{1}{N\sqrt{1-n^k n^k}} \begin{pmatrix} 1 & n^i \\ -n^j & -n^i n^j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X^{ij}(\gamma, n) \end{pmatrix}$$

- We see that, after this redefinition, the quantity $N\,V$ in the action is linear in the lapse, and therefore one of the constraints remains, leaving 5 degrees of freedom
- One can show that the secondary constraint is also there, and the analysis can be extended to the general ghost-free potentials and to bimetric versions too.

Hassan, Rosen; 1106.3344 1109.3515 1111.2070

A simple approach

- As a simple way to perform this analysis, we have proposed to introduce the square root implicitly

$$N\,V = 2\mathrm{m}^2 \Phi^{\mu}_{\mu} + \kappa^{\mu}_{\nu}\left(\Phi^{\nu}_{\alpha}\Phi^{\alpha}_{\mu} - N^2 g^{\nu\alpha}\eta_{\alpha\mu}\right)$$

or, after integrating out the Lagrange multiplier,

$$N\,V = m^2\left(\Phi^{\mu}_{\mu} + \left(\Phi^{-1}\right)^{\mu}_{\nu} N^2 g^{\nu\alpha}\eta_{\alpha\mu}\right)$$

- Now the constraint analysis can be done without explicitly taking any square roots.
- Intuitively one can better see that something interesting is there by defining $\Phi^2 = g^{-1}\eta$ instead of $\Phi^2 = N^2 g^{-1}\eta$, as it would produce a constraint $\left(\Phi^{-1}\right)^i_j \gamma^{ij} = 0$ with no explicit lapse or shifts in it.
- Golovnev, Phys.Lett.B **707** (2012), 404-408; 1112.2134

Vielbein formalism

- There is also a very interesting vielbein (vierbein) formulation of the model (Hinterbichler, Rosen; 1203.5783)
- A posteriori, it is very natural to work with vielbeins when the square roots of metric quantities are needed
- Moreover, this approach allows to tackle the multimetric theories. The ghost free potentials are all possible wedge products of the vielbeins of the metrics in the model – a very nice and beautiful result!
- The multimetric theories are not so easy to handle in the metric approach, therefore it would be important to analyse further this work, and to establish connections between different approaches to the Hamiltonian analysis of ghost-free bimetric and multimetric gravities.

Conclusions

- It is not so easy to consistently modify the GR
- Remarkably, a 3-parameter family of self-consistent massive deformations has been proposed recently
- Non-perturbative analysis completely confirms the perturbative claims of absence of the BD ghost
- The model is quite involved and refers to a matrix square root, but the analysis still can be done
- There are also several ways to proceed which offer some important simplifications
- The theory is really worth studying further. There is already a vast number of papers on foundations, on massive cosmologies, and on black hole solutions.