Nontrivial Kalb-Ramond field of the effective nongeometric background

Ljubica Davidović and Branislav Sazdović

Institute of Physics,
University of Belgrade,
www.ipb.ac.rs

MPHYS7
September 9-19, 2012
Beograd

Outline

- The open bosonic string in the weakly curved background
- How to solve the bc?
- First possibility:
  Treat the boundary conditions as constraints
  Phys.Rev.**D 83** (2011) 066014, JHEP**08** (2011) 112
- Second possibility:
  Start looking for the solution right away, guessing its appropriate form
- Exploring the second possibility
- Effective theory on the solution
- Nongeometrical background

Action

- describing the open bosonic string

$$S = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x)\right] \partial_\alpha x^\mu \partial_\beta x^\nu, \quad (\varepsilon^{01} = -1)$$

given in the conformal gauge $\qquad g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$.

- space-time equations of motion must be satisfied to have conformaly invariant theory on the quantum level

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0, \qquad D_\rho B^\rho{}_{\mu\nu} = 0$$

- Weakly curved background

$$G_{\mu\nu}(x) = const$$

$$B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho,$$

$$b_{\mu\nu} = const, \quad B_{\mu\nu\rho} = infinitesimal\ const$$

Equation of motion and boundary conditions

- Equation of motion $\quad \ddot{x}^\mu = x''^\mu - 2B^\mu{}_{\nu\rho}\dot{x}^\nu x'^\rho$
- Boundary conditions
$\gamma_\mu^0\Big|_{\sigma=0,\pi} \equiv \frac{\delta\mathcal{L}}{\delta x'^\mu}\Big|_{\sigma=0,\pi} = \Big[G_{\mu\nu}x'^\nu - 2B_{\mu\nu}\dot{x}^\nu\Big]\Big|_{\sigma=0,\pi} = 0$
- Only the even part of γ_μ^0 contributes to the boundary condition at $\sigma = 0$
- Introducing the even and odd coordinate parts with respect to $\sigma = 0$

$$q^\mu(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu}\Big|_{\sigma=0}, \quad \bar{q}^\mu(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu}\Big|_{\sigma=0}$$

- the boundary condition at $\sigma = 0$ becomes

$$\gamma_\mu^0 \equiv G_{\mu\nu}\bar{q}'^\nu - 2b_{\mu\nu}\dot{q}^\nu - 2h_{\mu\nu}(q)\dot{q}^\nu - 2h_{\mu\nu}(\bar{q})\dot{\bar{q}}^\nu$$

where $h_{\mu\nu}(x) = B_{\mu\nu}(x) - b_{\mu\nu}$ is infinitesimally small

How to solve boundary condition?

- The zeroth order
  - BC $\gamma_\mu^0 \equiv G_{\mu\nu}\bar{q}'^\nu - 2b_{\mu\nu}\dot{q}^\nu.$
  - Solution of BC and consistency condition $(\dot{\bar{q}}^\mu)' = (\bar{q}'^\mu)^{\cdot}$

$$\dot{\bar{q}}^\mu = 2b^\mu{}_\nu q'^\nu, \qquad \bar{q}'^\mu = 2b^\mu{}_\nu \dot{q}^\nu.$$

- We look for the solution of the BC, in the form

$$\begin{aligned} \dot{\bar{q}}^\mu &= -A^\mu_{1\nu}(\tilde{q})\dot{q}^\nu + 2\beta^\mu_{1\nu}(q)q'^\nu, \\ \bar{q}'^\mu &= -A^\mu_{2\nu}(\tilde{q})q'^\nu + 2\beta^\mu_{2\nu}(q)\dot{q}^\nu. \end{aligned}$$

- $A^\mu_{1\nu}$ and $A^\mu_{2\nu}$ must be odd, $\beta^\mu_{1\nu}$ and $\beta^\mu_{2\nu}$ must be even functions
- Beside satisfying BC, the solution must obey EM

$$\begin{aligned} \ddot{q}^\mu - q''^\mu &= -2B^\mu{}_{\nu\rho}\Big[\dot{q}^\nu \bar{q}'^\rho + \dot{\bar{q}}^\nu q'^\rho\Big], \\ \ddot{\bar{q}}^\mu - \bar{q}''^\mu &= -2B^\mu{}_{\nu\rho}\Big[\dot{q}^\nu q'^\rho + \dot{\bar{q}}^\nu \bar{q}'^\rho\Big], \end{aligned}$$

$(\dot{\bar{q}}^\mu)' = (\bar{q}'^\mu)^{\cdot}$ and it must be in agreement with 0th solution.

How to solve boundary condition?

- From the symmetry of EM and CC follows
  $A^{\mu}_{1\nu} = A^{\mu}_{2\nu} \equiv A^{\mu}{}_{\nu}, \quad \beta^{\mu}_{1\nu} = \beta^{\mu}_{2\nu} \equiv \beta^{\mu}{}_{\nu}.$
- Substituting ansatz into BC for $\sigma = 0$

$$\begin{aligned}\gamma^{0}_{\mu}\Big|_{\sigma=0} = 0 \quad &= \quad \Big[G_{\mu\nu}[-A^{\nu}_{\rho}(\tilde{q})q'^{\rho} + 2\beta^{\nu}_{\rho}(q)\dot{q}^{\rho}] \\ &- \quad 2b_{\mu\nu}\dot{q}^{\nu} - 2h_{\mu\nu}(q)\dot{q}^{\nu} - 2h_{\mu\nu}(\bar{q})2b^{\nu}{}_{\rho}q'^{\rho}\Big]\Big|_{\sigma=0} \\ &\to \Big[G\beta(q)\Big]_{\mu\nu}\Big|_{\sigma=0} = B_{\mu\nu}(q)\Big|_{\sigma=0}\end{aligned}$$

  which gives $\beta^{\mu}{}_{\nu}(q) = (G^{-1})^{\mu\rho}B_{\rho\nu}(q)$.
- The ansatz becomes

$$\dot{\bar{q}}^{\mu} = -A^{\mu}{}_{\nu}(\tilde{q})\dot{q}^{\nu} + 2\Big[G^{-1}B(q)\Big]^{\mu}{}_{\nu}q'^{\nu},$$

$$\bar{q}'^{\mu} = -A^{\mu}{}_{\nu}(\tilde{q})q'^{\nu} + 2\Big[G^{-1}B(q)\Big]^{\mu}{}_{\nu}\dot{q}^{\nu}.$$

Unknown coefficient $A^{\mu}{}_{\nu}$

- Two equations from EM and consistency condition

$$\dot{A}^{\mu}{}_{\nu}(\tilde{q})\dot{q}^{\nu} - A'^{\mu}_{\nu}(\tilde{q})q'^{\nu} = 2h'^{\mu}_{\nu}\dot{q}^{\nu} - 24h'^{\mu}_{\nu}(bq)(b\dot{q})^{\nu}$$

$$\dot{A}^{\mu}{}_{\nu}(\tilde{q})q'^{\nu} - A'^{\mu}_{\nu}(\tilde{q})\dot{q}^{\nu} = 24\Big[b\dot{h}b\dot{q} - bh'bq'\Big]^{\mu}$$

- Parity argument $\dot{\tilde{q}}^{\mu} = q'^{\mu}, \quad \tilde{q}'^{\mu} = \dot{q}^{\mu}$
- $\dot{A}^{\mu}{}_{\nu}(\tilde{q}) = A^{\mu}{}_{\nu}(\dot{\tilde{q}}) = A^{\mu}{}_{\nu}(q'), \quad A'^{\mu}_{\nu}(\tilde{q}) = A^{\mu}{}_{\nu}(\tilde{q}') = A^{\mu}{}_{\nu}(\dot{q})$

The Solution for A

$$
\begin{aligned}
{}_{(1)}A^{\mu}{}_{\nu}(q) &= (G^{-1})^{\mu\rho}\Big[h(q) - 12h(bq)b\Big]_{\rho\nu} \\
{}_{(2)}A^{\mu}{}_{\nu}(q) &= (G^{-1})^{\mu\rho}\Big[-12bh(q)b + 12bh(bq)\Big]_{\rho\nu}
\end{aligned}
$$

Because the solution ${}_{(1)}A^{\mu}{}_{\nu}$ satisfies the homogeneous part of the second equation and the solution ${}_{(2)}A^{\mu}{}_{\nu}$ satisfies the homogeneous part of the first equation, the complete solution for $A^{\mu}{}_{\nu}$ is of the form

$$
A^{\mu}{}_{\nu}(q) = (G^{-1})^{\mu\rho}\Big[h(q) - 12bh(q)b - 12h(bq)b + 12bh(bq)\Big]_{\rho\nu},
$$

with the property $(GA)_{\mu\nu} = -(GA)_{\nu\mu}$.

The Solution of BC

- The solution of BC at $\sigma = 0$

$$\begin{aligned} \dot{x}^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]\dot{q}^\nu + 2[G^{-1}B(q)]^\mu{}_\nu q'^\nu \\ x'^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]q'^\nu + 2[G^{-1}B(q)]^\mu{}_\nu \dot{q}^\nu \end{aligned}$$

- The solution of BC at $\sigma = \pi$

$$\begin{aligned} \dot{x}^\mu(\sigma) &= \Big[\delta^\mu_\nu - A^\mu{}_\nu[{}^\star\tilde{q}(\pi-\sigma)]\Big]{}^\star\dot{q}^\nu(\pi-\sigma) \\ &+ 2\Big[G^{-1}B[{}^\star q(\pi-\sigma)]\Big]^\mu_\nu {}^\star q'^\nu(\pi-\sigma) \\ x'^\mu(\sigma) &= \Big[\delta^\mu_\nu - A^\mu{}_\nu[{}^\star\tilde{q}(\pi-\sigma)]\Big]{}^\star q'^\nu(\pi-\sigma) \\ &+ 2\Big[G^{-1}B[{}^\star q(\pi-\sigma)]\Big]^\mu_\nu {}^\star\dot{q}^\nu(\pi-\sigma) \end{aligned}$$

$${}^\star q^\mu(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu}\Big|_{\sigma=\pi}, \quad {}^\star\bar{q}^\mu(\sigma) = -\sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu}\Big|_{\sigma=}$$

The Solution of BC

Note that if

$$q^\mu(\sigma) = {}^\star q^\mu(\pi - \sigma), \qquad \bar{q}^\mu(\sigma) = {}^\star \bar{q}^\mu(\pi - \sigma),$$

then the solutions for the bc in 0 and π coincide, and from the relation above follows the 2π-periodicity of x^μ. So, if we extend the σ domain and demand $2\pi-$periodicity of the original variable $x^\mu(\sigma + 2\pi) = x^\mu(\sigma)$, the relation

$$\begin{aligned} \dot{x}^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]\dot{q}^\nu + 2[G^{-1}B(q)]^\mu{}_\nu q'^\nu \\ x'^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]q'^\nu + 2[G^{-1}B(q)]^\mu{}_\nu \dot{q}^\nu \end{aligned}$$

solves both constraints at $\sigma = 0$ and $\sigma = \pi$.
Note that the solution of the boundary condition does not depend on one effective variable only, but on the two variables q^μ and $\tilde{q}^\mu$.

Effective theory (theory obtained on the solution of BC)

The effective Lagrangian equals

$$\mathcal{L}^{eff} = \frac{\kappa}{2}\dot{q}^{\mu} G^{eff}_{\mu\nu}(q,\tilde{q})\dot{q}^{\nu} - \frac{\kappa}{2} q'^{\mu} G^{eff}_{\mu\nu}(q,\tilde{q}) q'^{\nu} + 2\kappa q'^{\mu} B^{eff}_{\mu\nu}(q,\tilde{q})\dot{q}^{\nu},$$

where

$$\begin{aligned} G^{eff}_{\mu\nu}(q,\tilde{q}) &= G^{E}_{\mu\nu}(q+2b\tilde{q}) + 4[b^2 A(\tilde{q}) - A(\tilde{q})b^2]_{\mu\nu}, \\ B^{eff}_{\mu\nu}(q,\tilde{q}) &= [h(2b\tilde{q}) + 4bh(2b\tilde{q})b]_{\mu\nu} - B_{\mu}{}^{\rho}(q) G^{E}_{\rho\nu}(q). \end{aligned}$$

The expression

$$G^{E}_{\mu\nu}(x) \equiv G_{\mu\nu} - 4B_{\mu\rho}(x)(G^{-1})^{\rho\sigma} B_{\sigma\nu}(x)$$

is the definition of the open string metric. In the weakly curved background (in the leading order in $B_{\mu\nu\rho}$) it is equal to

$$\begin{aligned} G^{E}_{\mu\nu} &= g_{\mu\nu} - 4(bh+hb)_{\mu\nu}, \\ g_{\mu\nu} &= G_{\mu\nu} - 4(bG^{-1}b)_{\mu\nu} = G^{E}_{\mu\nu}(0). \end{aligned}$$

Effective theory

Hereafter, we will consider the action $S^{eff} = \int d\tau \int_{-\pi}^{\pi} d\sigma \mathcal{L}^{eff}$. This will cause the disappearance of the term in the effective metric which depends on $\tilde{q}$ and the term in effective Kalb-Ramond field which depends on q

$$S^{eff} = \kappa \int_{\Sigma_1} d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G^{eff}_{\mu\nu}(q) + \epsilon^{\alpha\beta} B^{eff}_{\mu\nu}(2b\tilde{q})\right] \partial_\alpha q^\mu \partial_\beta q^\nu,$$

$$\begin{aligned} G^{eff}_{\mu\nu}(q) &= G^E_{\mu\nu}(q), \\ B^{eff}_{\mu\nu}(2b\tilde{q}) &= -\frac{\kappa}{2}[g\Delta\theta(2b\tilde{q})g]_{\mu\nu}, \end{aligned}$$

$\Delta\theta$ is the infinitesimal part of the non-commutativity parameter

$$\theta^{\mu\nu} = -\frac{2}{\kappa}\left[G_E^{-1} B G^{-1}\right]^{\mu\nu} = \theta_0^{\mu\nu} - \frac{2}{\kappa}\left[g^{-1}(h + 4bhb)g^{-1}\right]^{\mu\nu},$$

$g_{\mu\nu} = G^E_{\mu\nu}(0)$ and $\theta_0^{\mu\nu} = \theta^{\mu\nu}(0) = -\frac{2}{\kappa}\left[g^{-1} b G^{-1}\right]^{\mu\nu}$.

Effective theory

- Unexpected things in the effective theory:
  - the appearance of the non-trivial Kalb-Ramond field $B^{eff}_{\mu\nu}$,
  - its dependence on the coordinate $\tilde{q}^\mu$ instead on q^μ (essentially the cause of the first).
- The theory of unoriented closed string (effective theory for constant background) does not contain Kalb-Ramond field.
  - The effective Kalb-Ramond field appears in the effective action within the term $B^{eff}_{\mu\nu}\dot{q}^\mu q'^\nu$.
  - If the Kalb-Ramond field depends on the Ω-even variable, this term does not contribute.
- In WCB the effective Kalb-Ramond field depends on $\tilde{q}^\mu$ (Ω-odd).
- $B^{eff}_{\mu\nu}(2b\tilde{q})$ is proportional to $\tilde{q}^\mu$.
- The effective Kalb-Ramond field is odd under σ-parity transformation $\Omega B^{eff}_{\mu\nu}[2b\tilde{q}(\sigma)] = -B^{eff}_{\mu\nu}[2b\tilde{q}(\sigma)]$.
- The term $B^{eff}_{\mu\nu}(2b\tilde{q})\dot{q}^\mu q'^\nu$ is Ω-even, so it survives.

Nongeometric background

The background of the effective theory depends on two variables q^μ and $\tilde{q}^\mu$. The first is the even part of the initial coordinate and the second satisfies

$$\dot{\tilde{q}}^\mu = q'^\mu, \qquad \tilde{q}'^\mu = \dot{q}^\mu.$$

The variable $\tilde{q}^\mu$ appears as an argument of $B^{eff}_{\mu\nu}$ only, which is the infinitesimal of the first order. The zeroth order equation of motion for q^μ is just $\partial_+\partial_- q^\mu = 0$, with the solution

$$q^\mu(\sigma) = f^\mu(\sigma^+) + f^\mu(\sigma^-).$$

From $\partial_\pm f^\mu(\sigma^\mp) = 0$, one has $\dot{f}^\mu(\sigma^\pm) = \pm f'^\mu(\sigma^\pm)$. Therefore, $\dot{q}^\mu(\sigma) = f'^\mu(\sigma^+) - f'^\mu(\sigma^-)$, and consequently for both equations we obtain

$$\tilde{q}^\mu(\tau,\sigma) = f^\mu(\sigma^+) - f^\mu(\sigma^-),$$

which means that $\tilde{q}^\mu(\tau,\sigma)$ is T-dual mapping of the effective coordinate $q^\mu(\tau,\sigma)$.

Conclusion

- The solution of the Neumann boundary conditions on the open string endpoints in the weakly curved background, causes the nongeometric effective background
- This background depends both on the conventional effective coordinate q^μ and on its T-dual $\tilde{q}^\mu$
- The complete transition from the original theory to the effective theory consists of
  1. the transition from conventional to the *doubled geometry*

$$x^\mu \to q^\mu, \; \tilde{q}^\mu$$

  2. *the background field* transition

$$G_{\mu\nu} \to G^{eff}_{\mu\nu}(q), \qquad B_{\mu\nu}(x) \to B^{eff}_{\mu\nu}(2b\tilde{q}) \,.$$

- The appearance of the doubled target space allowed the string to see the effective background field $B^{eff}_{\mu\nu}$.

Effective backgrounds

Let us summarize the forms of the initial and the corresponding effective backgrounds in the following table.

Original theory		*Effective theory*	
Metric field $G_{\mu\nu}$	*Kalb-Ramond field* $B_{\mu\nu}$	*Metric field* $G^{eff}_{\mu\nu}$	*Kalb-Ramond field* $B^{eff}_{\mu\nu}$
$G_{\mu\nu}$	$b_{\mu\nu}$	$g_{\mu\nu} = G_{\mu\nu} - 4b_{\mu\rho}b^{\rho}{}_{\nu}$	0
$G_{\mu\nu}$	$\frac{1}{3}B_{\mu\nu\rho}x^{\rho}$	$G_{\mu\nu}$	0
$G_{\mu\nu}$	$b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^{\rho}$	$G_{\mu\nu} - 4B_{\mu\rho}(q)B^{\rho}{}_{\nu}(q)$	$-\frac{\kappa}{2}(g\Delta\theta(2b\tilde{q})g)_{\mu\nu}$