Renormalizability of a noncommutative gauge theory

Maja Burić, University of Belgrade

arXiv: 1203.3016, 1003.2284

$M \cap \Phi$ 7, September 2012, Belgrade

Outline

Intuitively, noncommutativity of coordinates introduces through uncertainty relations the minimal length and the maximal momentum.

This in principle gives a possibility of regularization: of classical solutions as one cannot reach the singularity; of quantum field theory as we have an UV cutt-off. How to realize this concretely?

Our motivation:

General: to study noncommutative geometry

Concrete: to understand grounds of renormalizability of the GW model
to study matrix regularizations and matrix geometries beyond U_N

The simplest noncommutative space is given by $[x^\mu, x^\nu] = i\theta^{\mu\nu}=$const.
The field theory on it can be obtained by introducing the Moyal product
$\chi(x)\star\phi(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu}\chi(x)\phi(x')|_{x'\to x}$ on the space of functions.

The action for the scalar field theory is then

$$\mathcal{S} = \int \frac{1}{2}\,\partial_\mu\phi\star\partial^\mu\phi + \frac{\mu_0^2}{2}\,\phi\star\phi + \frac{\lambda}{4!}\,\phi\star\phi\star\phi\star\phi.$$

Though the vertex gets regulated in the UV,

$$\iint dp\,dq\,dk\,dl\;\delta(p+q+k+l)\cos\frac{k\wedge p}{2}\cos\frac{l\wedge q}{2}\,\phi(p)\phi(q)\phi(k)\phi(l),$$

new divergences appear in the IR and the theory is not renormalizable.

However, modified scalar field model of Grosse and Wulkenhaar

$$\mathcal{S} = \int \frac{1}{2}\,\partial_\mu\phi \star \partial^\mu\phi + \frac{\mu_0^2}{2}\,\phi\star\phi + \frac{\Omega^2}{4}(x^\mu x_\mu)\star\phi\star\phi + \frac{\lambda}{4!}\,\phi\star\phi\star\phi\star\phi$$

is fully renormalizable.

Added is the oscillator potential term. It yields an extra symmetry (LS duality); it confines the scalar field; it breaks translation invariance.

A proof of renormalizability can be done in the matrix base, where coordinates x^μ are represented by $\infty\times\infty$ matrices: calculations are done for $n\times n$ matrices and then the $n\to\infty$ limit is taken.

Is this a matrix regularization of scalar field theory?

Questions

- What is the origin of renormalizability: Langmann-Szabo duality, confinement through the potential, or a specific matrix regularization?
- How to implement the idea to a gauge model that is, how to make it compatible with gauge symmetry?

There are various attempts to anwer these questions

- D. N. Blaschke, E. Kronberger, R. I. P. Sedmik and M. Wohlgenannt, [arXiv:1004.2127 [hep-th]].
- A. de Goursac, [arXiv:0710.1162 [hep-th]].
- H. Grosse and M. Wohlgenannt, [hep-th/0703169].
- A. de Goursac, J. C. Wallet and R. Wulkenhaar, [arXiv:0803.3035 [hep-th]].
- D. N. Blaschke, H. Grosse and M. Schweda, [arXiv:0705.4205 [hep-th]].
- D. N. Blaschke, H. Grosse, E. Kronberger, M. Schweda and M. Wohlgenannt, [arXiv:0912.3642 [hep-th]].
- D. N. Blaschke, F. Gieres, E. Kronberger, M. Schweda and M. Wohlgenannt, [arXiv:0804.1914 [hep-th]].
- D. N. Blaschke, A. Rofner, M. Schweda and R. I. P. Sedmik, [arXiv:0901.1681 [hep-th]].

A matrix base on the Moyal space is given through the basis of harmonic oscillator in which the position algebra $[x, y] = i$ is represented by infinite matrices

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 0 & \sqrt{n-1} & . \\ . & . & . & . & \sqrt{n-1} & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 0 & -\sqrt{n-1} & . \\ . & . & . & . & \sqrt{n-1} & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

We can truncate to finite $n \times n$ matrices. This changes the commutation relation to $[x, y] = i(1 - z)$

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & \sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

$$y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & -\sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

$$z = n \begin{pmatrix} 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & 0 \\ . & . & . & . & 0 & 1 \end{pmatrix}$$

or more precisely, to a quadratic algebra. Introducing mass parameters μ and $\bar{\mu}$ and dimensionless ϵ, we define the truncated Heisenberg algebra:

$$[\mu x, \mu y] = i\epsilon(1 - \bar{\mu} z)$$

$$[\mu x, \bar{\mu} z] = i\epsilon(\mu y\, \bar{\mu} z + \bar{\mu} z\, \mu y)$$

$$[\mu y, \bar{\mu} z] = -i\epsilon(\mu x\, \bar{\mu} z + \bar{\mu} z\, \mu x).$$

It can be considered as a 3-dimensional noncommutative space.

- $\epsilon = 0$ is the 'commutative limit'
- $\bar{\mu} = 0$ is a contraction to the Heisenberg algebra; on the level of representations, it is a weak limit. Alternatively, it is a 2-dimensional 'subspace' $z = 0$
- for $\epsilon = 1$, $\bar{\mu} = \mu$ the algebra has finite representations

Every finite matrix algebra is a smooth noncommutative space, that is, it can be endowed with diffferential structure. But as $x^\mu x^\nu \neq x^\nu x^\mu$, we also have

$$x^\mu dx^\nu \neq dx^\nu x^\mu$$
$$dx^\mu dx^\nu \neq -dx^\nu dx^\mu .$$

The last two commutation rules do not follow automatically from the commutator $[x^\mu, x^\nu]$ but have to be defined.

The key ingredient in the matrix case is that derivations = commutators with momenta:

$$e_\alpha \phi = [p_\alpha, \phi].$$

These derivations are inner: momenta belong to the same matrix algebra.

The first ingredient of geometry are derivations e_α that is momenta p_α.

The second ingredient are dual frame 1-forms θ^α: $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$. We assume that $[\phi, \theta^\alpha] = 0$.

The differential d is defined as $d\phi = (e_\alpha \phi)\, \theta^\alpha$.

The next assumption is that metric is constant in the frame basis, $g^{\alpha\beta} = g(\theta^\alpha \otimes \theta^\beta) = \eta^{\alpha\beta}$.

In addition, all structures are assumed to be linear.

One can proceed quite straightforwardly and define linear connection $\omega^\alpha{}_\beta = \omega^\alpha{}_{\gamma\beta}\theta^\gamma$, curvature $\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \omega^\gamma{}_\beta$ and so on.

However, constraints as $d^2 = 0$ or $d\,[\phi, \theta^\alpha] = 0$ are not automatically satisfied and have to be imposed separately: this makes the construction somewhat rigid. Hermiticity is also nontrivial.

For the truncated Heisenberg algebra we introduce momenta as

$$\epsilon p_1 = i\mu^2 y, \quad \epsilon p_2 = -i\mu^2 x, \quad \epsilon p_3 = i\mu(\mu z - \frac{1}{2})$$

and thus we obtain a differential structure. Momenta p_1 and p_2 are the same as in the Heisenberg algebra. We fix the linear connection and calculate the curvature:

$$\text{connection}: \quad \begin{aligned} \omega_{12} &= -\omega_{21} = \mu\,(\frac{1}{2} - 2\mu z)\,\theta^3 \\ \omega_{13} &= -\omega_{31} = \frac{\mu}{2}\,\theta^2 + 2\mu^2 x\,\theta^3 \\ \omega_{23} &= -\omega_{32} = -\frac{\mu}{2}\,\theta^1 + 2\mu^2 y\,\theta^3 \end{aligned}$$

$$\text{curvature scalar:} \quad R = \tfrac{11}{4}\mu^2 - 2\mu^2(\mu z - \tfrac{1}{2}) - 4\mu^4(x^2 + y^2)$$

Scalar field

Since the curvature scalar is quadratic, one recognizes easily that the GW action can be understood as the action for scalar field on a curved space

$$S = \int \frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi + \frac{M^2}{2}\phi^2 - \frac{\xi}{2}\,R\phi^2 + \frac{\Lambda}{4!}\,\phi^4,$$

after dimensional reduction to $z = 0$.

This is nice because it indicates that perhaps the GW model is geometric, and that the geometry involved is that of a finite matrix space. Is it possible to obtain a gauge theory too? What are its renormalization properties?

Gauge fields

The noncommutative U_1 gauge fields in this formalism are defined as on a commutative space. The vector potential is a 1-form, the field strength is a 2-form:

$$A = A_\alpha \theta^\alpha, \qquad F = dA + A^2 = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta.$$

But as 1-forms are noncommuting, $\{\theta^\alpha, \theta^\beta\} = 2i\epsilon Q^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta$, we have

$$F_{\alpha\beta} = e_{[\alpha}A_{\beta]} - A_\gamma C^\gamma{}_{\alpha\beta} + [A_\alpha, A_\beta] + 2i\epsilon(e_\delta A_\gamma)Q^{\delta\gamma}{}_{\alpha\beta} + 2i\epsilon A_\delta A_\gamma Q^{\delta\gamma}{}_{\alpha\beta}$$

$Q^{\alpha\beta}{}_{\gamma\delta} \neq 0$ only when the momentum algebra is quadratic.

The Yang-Mills action is (with additional details which depend on the definition of volume form and Hodge dual)

$$\mathcal{S}_{YM} = \frac{1}{4}\int F_{\alpha\beta}F^{\alpha\beta}.$$

When calculus is based on inner derivations there is a special 1-form θ, $\theta = -p_\alpha \theta^\alpha$ invariant under the gauge group. The difference between A and θ, $X_\alpha = p_\alpha + A_\alpha$ transforms in the adjoint representation: X_α are called the covariant coordinates.

The field strength expressed in covariant coordinates is

$$F_{\alpha\beta} = 2P^{\gamma\delta}{}_{\alpha\beta} X_\gamma X_\delta - F^\gamma{}_{\alpha\beta} X_\gamma - \frac{1}{i\epsilon} K_{\alpha\beta}.$$

In particular if momenta are the generators of a Lie group this becomes

$$F_{\alpha\beta} = [X_\alpha, X_\beta] - C^\gamma{}_{\alpha\beta} X_\gamma$$

and the Yang-Mills action obtains a form known from matrix models.

Reduced Yang-Mills action

On subspace $z = 0$ we have $p_3 = -\frac{i\mu}{2\epsilon}$, $e_3 = 0$ and A_3 transforms as a scalar field in the adjoint representation. We denote

$$\mathsf{A}_3 = \phi, \quad \mathsf{A}_1 = A_1, \quad \mathsf{A}_2 = A_2$$

Then

$$\mathsf{F}_{12} = F_{12} - \mu\phi = [\mathsf{X}_1, \mathsf{X}_2] + \frac{i\mu^2}{\epsilon} - \mu\phi$$
$$\mathsf{F}_{13} = D_1\phi - i\epsilon\{p_2 + A_2, \phi\} = [\mathsf{X}_1, \phi] - i\epsilon\{\mathsf{X}_2, \phi\}$$
$$\mathsf{F}_{23} = D_2\phi + i\epsilon\{p_1 + A_1, \phi\} = [\mathsf{X}_2, \phi] + i\epsilon\{\mathsf{X}_1, \phi\}.$$

We derive after dimensional reduction the Yang-Mills action of our model

$$\mathcal{S}_{YM} = \frac{1}{2}\,\mathrm{Tr}\,\big((1-\epsilon^2)(F_{12})^2 - 2(1-\epsilon^2)\mu F_{12}\phi + (5-\epsilon^2)\mu^2\phi^2 + 4i\epsilon F_{12}\phi^2$$
$$+ (D_1\phi)^2 + (D_2\phi)^2 - \epsilon^2\{p_1 + A_1, \phi\}^2 - \epsilon^2\{p_2 + A_2, \phi\}^2\big).$$

The classical YM equations of motion have two simple vacua:

$$A_1 = 0,\ A_2 = 0,\ \phi = 0,$$
$$\mathrm{X}_1 = 0,\ \mathrm{X}_2 = 0,\ \mathrm{X}_3 = 0.$$

The first can be used in quantization; the second vacuum is stable when we add the Chern-Simons action:

$$\mathcal{S}_{CS} = \int \mathrm{X}^3 = \int (3-\epsilon^2)(F_{12} - \frac{i\mu^2}{\epsilon})\phi + \frac{2i\epsilon}{3}((p_1 + A_1)^2 + (p_2 + A_2)^2)(\phi - \frac{i\mu}{2\epsilon})$$

The BRST invariance of the quantum action with the gauge fixing term of the form $\mathcal{G} = e_\alpha A^\alpha = \partial_1 A^1 + \partial_2 A^2$ can be straightforwardly verified.

We fix the gauge and quantize the Yang-Mills reduced action, using as vacuum the trivial one, $A_\alpha = 0$, $\phi = 0$.

The action contains two interacting fields: the scalar field ϕ is of the GW type; its mixing with the gauge field A_α occurs even in the kinetic part.

Denoting $a = 1 - \epsilon^2$ we have:

Gauge fixed action

Kinetic term

$$\mathcal{S}_{kin} = -\frac{1}{2}\int \; aA_\alpha \Box A^\alpha + 2a\mu\epsilon^{\alpha\beta}(\partial_\alpha A_\beta)\phi$$

$$+\,\phi\Box\phi - (4+a)\mu^2\phi^2 - 4\mu^4 x^\alpha x_\alpha \phi^2 \;\; + 2\bar{c}\,\Box c$$

Interaction

$$\mathcal{S}_{int} = -\frac{1}{2}\int \; 4\epsilon\epsilon_{\alpha\beta}(\partial^\alpha A^\beta + iA^\alpha \star A^\beta)\star\phi^2 - 2i(\partial_\alpha\phi)[A^\alpha \overset{\star}{,} \phi]$$

$$+2ia\mu\epsilon_{\alpha\beta}A^\alpha \star A^\beta\phi - 2ia\epsilon_{\alpha\beta}\partial^\alpha A^\beta\epsilon_{\gamma\delta}A^\gamma \star A^\delta + a(\epsilon_{\alpha\beta}A^\alpha \star A^\beta)^2$$

$$+[A_\alpha \overset{\star}{,} \phi][A^\alpha \overset{\star}{,} \phi] - \epsilon^2\{A_\alpha \overset{\star}{,} \phi\}\{A^\alpha \overset{\star}{,} \phi\} + 2\mu^2\epsilon\epsilon_{\alpha\beta}\{x^\alpha \overset{\star}{,} \phi\}\{A^\beta \overset{\star}{,} \phi\}$$

$$-i\bar{c}\partial_\alpha[A^\alpha \overset{\star}{,} c]$$

To obtain the propagator we treat A_α and ϕ as doublet of fields

$$\mathcal{S}'_{kin} = -\frac{1}{2}\int \begin{pmatrix} A^\mu & \phi \end{pmatrix} \begin{pmatrix} a\Box\delta_{\mu\nu} & -a\mu\epsilon_{\mu\zeta}\partial^\zeta \\ a\mu\epsilon_{\nu\eta}\partial^\eta & K^{-1} - a\mu^2 \end{pmatrix} \begin{pmatrix} A^\nu \\ \phi \end{pmatrix}$$

and for the kinetic operator we get

$$G = \begin{pmatrix} \frac{1}{a}\Box^{-1}\delta_{\mu\nu} - \mu^2\Box^{-1}\epsilon_{\mu\zeta}\partial^\zeta K \epsilon_{\nu\eta}\partial^\eta\Box^{-1} & -\mu\Box^{-1}\epsilon_{\mu\zeta}\partial^\zeta K \\ \mu K \epsilon_{\nu\eta}\partial^\eta\Box^{-1} & K \end{pmatrix}$$

where K is the so-called Mehler kernel, the propagator of the scalar field in the harmonic oscillator potential.

Mehler kernel

By definition

$$K^{-1}(x,y) = (\Box - 4\mu^4 x_\alpha x^\alpha - 4\mu^2)\delta^2(x-y).$$

The inverse can be found explicitly in the parameter form in position and in momentum space. In 2 dimensions

$$K(p,q) = -\frac{\pi}{4\mu^4}\int_0^\infty \frac{\omega\, d\tau}{\sinh\omega\tau}\, e^{-\frac{1}{8\mu^2}\left((p+q)^2\coth\frac{\omega\tau}{2} + (p-q)^2\tanh\frac{\omega\tau}{2}\right) - \omega\tau}$$

where the last term in the exponent is related to the mass of the field; we have $\omega\tau = (\mu_0^2/4\mu^2)\,\omega\tau$.

When one introduces dimensionless parameter $\alpha = \omega\tau$ or $\xi = \coth\frac{\alpha}{2}$

$$K(p,q) = -\frac{\pi}{4\mu^4}\int_1^\infty \frac{d\xi}{\xi}\,\frac{\xi-1}{\xi+1}\, e^{-\frac{1}{8\mu^2}\left((p+q)^2\xi + (p-q)^2\frac{1}{\xi}\right)}.$$

Clearly the translation invariance is broken as we have dependence on both $p+q$ and $p-q$; in the limit $\omega \to 0$:

$$K(p,q)|_{\omega\to 0} = -\frac{(2\pi)^2}{p^2+\mu_0^2}\,\delta^2(p+q).$$

Field contractions are given by

$$\underbrace{\phi(k)\phi(l)} \equiv K(k,l),$$

$$\underbrace{A_\sigma(k)\phi(l)} = -i\mu\,\frac{\epsilon_{\sigma\beta}k^\beta}{k^2}\,K(k,l),$$

$$\underbrace{\phi(k)A_\sigma(l)} = -i\mu K(k,l)\,\frac{\epsilon_{\sigma\beta}l^\beta}{l^2},$$

$$\underbrace{A_\rho(k)A_\sigma(l)} = -\frac{(2\pi)^2}{a}\frac{\delta_{\rho\sigma}}{k^2}\,\delta(k+l) + (-i\mu)^2\,\frac{\epsilon_{\rho\nu}k^\nu}{k^2}\,K(k,l)\,\frac{\epsilon_{\sigma\tau}l^\tau}{l^2}\,,$$

$$\underbrace{\bar c(k)c(l)} = -\frac{(2\pi)^2}{k^2}\,\delta(k+l)\,.$$

Vertices

One can define Feynman rules for vertices straigforwardly; we have ten of them:

$$(1)\quad \frac{2i\epsilon}{(2\pi)^4}\int dp\,dq\,dk\,\delta(p+q+k)\cos\frac{k\wedge q}{2}\,\epsilon_{\rho\sigma}p^\rho A^\sigma(p)\phi(q)\phi(k)$$

$$(2)\quad \frac{2i}{(2\pi)^4}\int dp\,dq\,dk\,\delta(p+q+k)\sin\frac{q\wedge k}{2}\,p^\rho\phi(p)A_\rho(k)\phi(q)$$

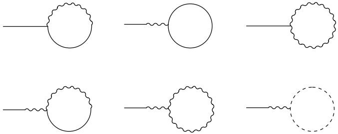
$$(3)\quad \frac{-4i\mu^2\epsilon}{(2\pi)^4}\int dp\,dq\,dk\,\delta(p+q+k)\cos\frac{k\wedge q}{2}\epsilon_{\rho\sigma}\frac{\partial\phi(p)}{\partial p_\sigma}A^\rho(k)\phi(q)$$

$$\vdots$$

$$(9)\quad \frac{2\epsilon}{(2\pi)^6}\int dp\,dq\,dk\,dl\,\delta(p+q+k+l)\sin\frac{q\wedge p}{2}\cos\frac{l\wedge k}{2}\,\epsilon^{\rho\sigma}A_\rho(p)A_\sigma(q)\phi(k)\phi(l)$$

$$(10)\quad \frac{a}{2(2\pi)^6}\int dp\,dq\,dk\,dl\,\delta(p+q+k+l)\sin\frac{q\wedge p}{2}\sin\frac{l\wedge k}{2}\epsilon^{\rho\sigma}A_\rho(p)A_\sigma(q)\epsilon^{\lambda\tau}A_\lambda(k)A_\tau(l).$$

Tadpoles $T(r) = -\langle \phi(r)\, \mathcal{S}_{int} \rangle$ and $T_\mu(r) = -\langle A_\mu(r)\, \mathcal{S}_{int} \rangle$ do not vanish. The corresponding diagrams are



$$T_\nu(r) = -i\mu\, \frac{\tilde{r}_\nu}{r^2}\, T(r) + \mathbb{B}_\nu(r)$$

$$T(r) = \frac{2\mu}{(2\pi)^4} \int dp\, dq\, dk\, \delta(p+q+k)$$

$$\times \left(\epsilon \cos \frac{p \wedge q}{2} \,(1 + 2\mu^2 \frac{p_\sigma}{p^2}\, \frac{\partial}{\partial q_\sigma}) + \sin \frac{q \wedge p}{2}\, \frac{p \cdot \tilde{q}}{p^2 q^2}\right) \mathcal{K}(r,p,q,k),$$

with

$$\mathbb{B}_\mu(r) = \frac{i}{(2\pi)^2 a} \int dp\, dq\, \delta(p+q-r)$$
$$\times \Big(2\epsilon \cos\frac{p\wedge q}{2} \,\frac{\epsilon_{\mu\alpha}}{r^2}(r^\alpha - 2\mu^2 \frac{\partial}{\partial p_\alpha}) + \sin\frac{q\wedge p}{2}\,(\frac{2p_\mu}{r^2} + a\mu^2 \frac{\tilde r_\mu}{r^2}\,\frac{p\cdot\tilde q}{p^2 q^2})\Big)\, K(p,q)$$

We introduced the cyclic product of two Mehler kernels

$$\mathcal{K}(r,p,q,k) = K(r,p)K(q,k) + K(r,q)K(p,k) + K(r,k)K(p,q).$$

There are two nontrivial momentum and two parameter integrations in the final expressions, relatively difficult to analyze.

To extract divergences we amputate the external leg

$$\begin{pmatrix} \tau_\mu(s) \\ \tau(s) \end{pmatrix} = \frac{1}{(2\pi)^2} \int dr\, G^{-1}(s,-r) \begin{pmatrix} T_\nu(r) \\ T(r) \end{pmatrix}$$

and calculate the divergence. We obtain

$$\tau_\mu(s) = 4i\mu^2 \frac{\tilde{s}_\mu}{s^4}\, e^{-\frac{s^2}{4\mu^2}}$$

$$\tau(s) = -\frac{1}{\mu}\, e^{\frac{s^2}{4\mu^2}} \Big(E_0(\frac{s^2}{2\mu^2}) - E_1(\frac{s^2}{2\mu^2}) \Big)$$

The tadpole is regular in the UV and singular in the IR; E_n are exponential integrals.

To obtain the corresponding terms in the effective action we calculate

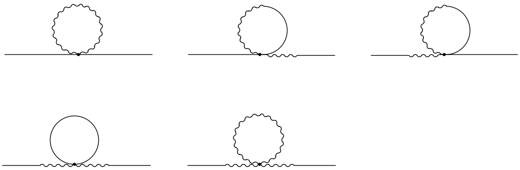
$$\int ds\, \tau_\mu(s) A^\mu(s), \qquad \int ds\, \tau(s)\phi(s)$$

and extract the divergent part. Expanding ϕ and A_μ in Taylor series to find divergence we obtain only two infinite contributions at one loop:

$$\int dx\, \phi, \qquad \int dx\, \epsilon_{\alpha\beta}\, x^\alpha \star A^\beta.$$

Divergences are logarithmic.

Propagator corrections are given by



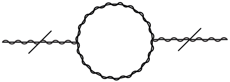
$$P_{\phi(r)\phi(s)} \equiv P(r,s) = -\langle \phi(r)\phi(s)\mathcal{S}_{int}\rangle$$

$$P_{\phi(r)A_\mu(s)} \equiv P_\mu(r,\mu s) = P_\mu(\mu s, r) = -\langle \phi(r)A_\mu(s)\mathcal{S}_{int}\rangle$$

$$P_{A_\nu(r)A_\mu(s)} \equiv P_{\nu\mu}(\nu r,\mu s) = -\langle A_\nu(r)A_\mu(s)\mathcal{S}_{int}\rangle.$$

In a similar way we calculate one loop contributions to propagators, everything is just much longer.

As before we define amputated graphs $\Pi = G^{-1}PG^{-1}$



Using the amputated graphs we calculate the divergences in the effective action, that is counterterms $\int \phi\Pi\phi$, $\int \phi\Pi_\mu A^\mu$, etc.

For the amputated 2-point function $\Pi_\rho(r,s)$ for example we obtain

$$\frac{2i\mu}{(2\pi)^2}\int dk\, dl\, \delta(k+l-r-s)\, \cos\frac{s\wedge l}{2}\, K(k,l)\big(\frac{4\tilde{k}_\rho}{k^2}\, \cos\frac{r\wedge k}{2} - \frac{k_\rho}{k^2}\, \sin\frac{r\wedge k}{2}\big)$$

$$= \frac{i(2\pi)^4}{4\mu}\,\frac{1}{u^2}\int_1^\infty \frac{d\xi}{(\xi+1)^2}\, e^{-\frac{u^2}{4\mu^2}\,\xi}\Big((4-\xi)\tilde{u}_\rho\cos\frac{u\wedge v}{4} + (4\xi-1)u_\rho\sin\frac{u\wedge v}{4}\Big)$$

$$-\frac{i(2\pi)^4}{4\mu}\int_1^\infty d\xi\,\frac{\xi-1}{\xi+1}\, e^{-\frac{1}{8\mu^2}\,(u^2+v^2)\xi}\,\frac{1}{(u-\xi v)^2}\,\frac{1}{(u-\xi v)^2}$$

$$\times\Big(\cos\frac{u\wedge v}{2}\,\big((4\tilde{u}_\rho-\xi\tilde{v}_\rho)(u^2-\xi^2v^2)+2(u_\rho-4\xi v_\rho)u\cdot\xi\tilde{v}\big)$$

$$+\sin\frac{u\wedge v}{2}\,\big((u_\rho-4\xi v_\rho)(u^2-\xi^2v^2)-2(4\tilde{u}_\rho-\xi v_\rho)u\cdot\xi\tilde{v}\big)\Big)$$

where $u=r+s$, $v=r-s$ are the so-called ‘short’ and ‘long’ variable.

We obtain that only three terms in the effective action which are, again logarithmically, divergent. They are

$$\int dx\, A_\alpha \star A^\alpha, \qquad \int dx\, \phi \star \phi, \qquad \int dx\, \epsilon_{\alpha\beta} \{x^\alpha \overset{\star}{,} A^\beta\} \star \phi.$$

This result has to be completed with one-loop second-order propagator corrections.

- Some counterterms which we obtained are not present in the classical action. Notably, the tadpoles do not vanish.
- There are two ways to understand these terms. One possibility is that the trivial vacuum $\phi = 0$, $A_\alpha = 0$ is unstable under quantization, and that the quantum vacuum has the form

$$\phi = \alpha, \qquad A_\alpha = \beta \epsilon_{\alpha\beta} x^\beta$$

  analogous to the second of the classical vacua.
- The second possibility is that all counterterms add up to Chern-Simons action.
- Most likely, divergences indicate that the theory has to be fully LS dual in order to be renormalizable.

Outlook

- Corrections to vertices: this will help to decide whether the origin of divergences is a shift of the vacuum or the Chern-Simons term.
- A systematic way to quantify divergences in the parameter integrals.

- If the model proves renormalizable, this result will mean that the correct regularization is matrix & geometrical: defined using the geometry of the underlying matrix space.
- The additional symmetry, Langmann-Szabo duality, seems to play a special role.