Some representations of planar Galilean conformal algebra

Naruhiko Aizawa

Osaka Prefecture University, Japan

Belgrade, 14.09.2012

Galilean conformal algebra

- a class of non-semisimple Lie algebras
- finite dim version / infinite dim version
- conformal algebra in nonrelativistic physical systems
- two parameters (d, ℓ)

$$d = 1, 2, \cdots (\text{dim of space}), \qquad \ell = 1/2, 1, 3/2, \cdots (\text{spin})$$

fix (d, ℓ) $\Rightarrow$ one algebra is defined

e.g. $\ell = 1/2$

- finite dim version $\Rightarrow$ Schrödinger algebra
- infinite dim version $\Rightarrow$ Schrödinger-Virasoro algebra

C. Roger, J. Unterberger, *The Schrödinger-Virasoro Algebra: Mathematical structure and dynamical Schrödinger symmetries*, Springer, 2012

Today's topic

Representation theory of $d = 2$(planar), $\ell = 1$ infinite dimensional algebra

- highest weight module
- coadjoint representation $\cdots$ coadjoint orbit

Today's topic

Representation theory of $d = 2$(planar), $\ell = 1$ infinite dimensional algebra

- highest weight module
- coadjoint representation $\cdots$ coadjoint orbit

Why $d = 2$?

- observations on $d = 2$ finite dim algebra
  - existence of exotic central extension (only for $d = 2$)
  - physical relevance
    non-rel AdS/CFT Martelli, Tachikawa 2010 ,
    classical mechanics Lukierski, Stichel, Zakrzewski 2006 2007
    Galilean electromagnetism Negro, del Olmo, Rodríguez-Marco 1997
  - representation theory $\cdots$ technique similar to semisimple case

    NA, Isaac 2011
- Virasoro algebra $\subset$ infinite dim algebra

$\Longrightarrow$ ∞ dim algebra may be of physical/mathematical interests

Outline

1. finite dim planar Galilean conformal algebra
   definition / central extension
2. infinite dim planar Galilean conformal algebra
   definition / central extension
3. highest weight representations (Verma modules)
   Verma modules / Kac determinant / irreducibility of VM
4. coadjoint representation
   coadjoint reps of infinite dim algebra and group / example of coadjoint orbit
5. summary

Outline

1. finite dim planar Galilean conformal algebra
   definition / central extension
2. infinite dim planar Galilean conformal algebra
   definition / central extension
3. highest weight representations (Verma modules)
   Verma modules / Kac determinant / irreducibility of VM
4. coadjoint representation
   coadjoint reps of infinite dim algebra and group / example of coadjoint orbit
5. summary

Finite dim algebra ($d = 2,\ \ell = 1$)

3 sets of generators : transformation in $(2+1)$ dim spacetime (t, x_1, x_2)

1. Galilei algebra $(i = 1, 2)$

$$H = \frac{\partial}{\partial t}, \quad P_0^i = \frac{\partial}{\partial x_i}, \quad P_1^i = -t\frac{\partial}{\partial x_i}, \quad J = -x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}$$

2. conformal transformation + dilatation

$$C = t^2\frac{\partial}{\partial t} + 2tx_i\frac{\partial}{\partial x_i}, \qquad D = -t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i}$$

$$(t, x_i) \xrightarrow{C} \left(\frac{t}{1-at}, \frac{x_i}{(1-at)^2}\right), \qquad (t, x_i) \xrightarrow{D} (at, ax_i)$$

3. transformation to accelerated frame $P_2^i = (-t)^2\dfrac{\partial}{\partial x_i}$

Finite dim algebra ($d = 2,\ \ell = 1$)

3 sets of generators : transformation in $(2+1)$ dim spacetime (t, x_1, x_2)

1. Galilei algebra $(i = 1, 2)$

$$H = \frac{\partial}{\partial t}, \quad P_0^i = \frac{\partial}{\partial x_i}, \quad P_1^i = -t\frac{\partial}{\partial x_i}, \quad J = -x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}$$

2. conformal transformation + dilatation

$$C = t^2\frac{\partial}{\partial t} + 2tx_i\frac{\partial}{\partial x_i}, \qquad D = -t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i}$$

$$(t, x_i) \xrightarrow{C} \left(\frac{t}{1-at}, \frac{x_i}{(1-at)^2}\right), \qquad (t, x_i) \xrightarrow{D} (at, ax_i)$$

3. transformation to accelerated frame $P_2^i = (-t)^2\frac{\partial}{\partial x_i}$

structure : $sl(2, \mathbb{R})$

Finite dim algebra ($d = 2,\ \ell = 1$)

3 sets of generators : transformation in $(2+1)$ dim spacetime (t, x_1, x_2)

1. Galilei algebra $(i = 1, 2)$

$$H = \frac{\partial}{\partial t}, \quad P_0^i = \frac{\partial}{\partial x_i}, \quad P_1^i = -t\frac{\partial}{\partial x_i}, \quad J = -x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}$$

2. conformal transformation + dilatation

$$C = t^2\frac{\partial}{\partial t} + 2tx_i\frac{\partial}{\partial x_i}, \qquad D = -t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i}$$

$$(t, x_i) \xrightarrow{C} \left(\frac{t}{1-at}, \frac{x_i}{(1-at)^2}\right), \qquad (t, x_i) \xrightarrow{D} (at, ax_i)$$

3. transformation to accelerated frame $P_2^i = (-t)^2\frac{\partial}{\partial x_i}$

structure : $sl(2,\mathbb{R}) \oplus so(2)$

Finite dim algebra ($d = 2,\ \ell = 1$)

3 sets of generators : transformation in $(2+1)$ dim spacetime (t, x_1, x_2)

1. Galilei algebra $(i = 1, 2)$

$$H = \frac{\partial}{\partial t}, \quad P_0^i = \frac{\partial}{\partial x_i}, \quad P_1^i = -t\frac{\partial}{\partial x_i}, \quad J = -x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}$$

2. conformal transformation + dilatation

$$C = t^2\frac{\partial}{\partial t} + 2tx_i\frac{\partial}{\partial x_i}, \qquad D = -t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i}$$

$$(t, x_i) \xrightarrow{C} \left(\frac{t}{1-at}, \frac{x_i}{(1-at)^2}\right), \qquad (t, x_i) \xrightarrow{D} (at, ax_i)$$

3. transformation to accelerated frame $P_2^i = (-t)^2\frac{\partial}{\partial x_i}$

structure : $sl(2,\mathbb{R}) \oplus so(2) \ni$ Abelian ideal

Finite dim algebra ($d = 2,\ \ell = 1$)

central extension

$\langle P_0^i, P_1^i, P_2^i \rangle_{i=1,2}$ Abelian $\Rightarrow$ non-Abelian

$$[P_m^i, P_n^j] = I_{mn}\varepsilon^{ij}\Theta, \quad \varepsilon^{12} = -\varepsilon^{21} = 1,\ I_{mn} = I_{nm}$$

only for $d = 2$ Stichel, Zakrzewski 2004, Lukierski, Stichel, Zakrzewski 2006 2007

$(\because)$ addition of this dose not contradict with Jacobi identity

Outline

1. finite dim planar Galilean conformal algebra
   definition / central extension
2. infinite dim planar Galilean conformal algebra
   definition / central extension
3. highest weight representations (Verma modules)
   Verma modules / Kac determinant / irreducibility of VM
4. coadjoint representation
   coadjoint reps of infinite dim algebra and group / example of coadjoint orbit
5. summary

Infinite dim extension

finite dim algebra (centerless) $\Rightarrow$ easily extend to ∞ dim algebra

Martelli, Tachikawa 2010

extension of vector field representation

finite dim $m = 0, 1, 2$	∞ dim $m \in \mathbb{Z}$
$H = \partial_t$	
$D = -t\partial_t - x_i\partial_{x_i}$	$L^m = -t^{m+1}\partial_t - (m+1)t^m x_i\partial_i$
$C = t^2\partial_t + 2tx_i\partial_{x_i}$	
$P^i_m = (-t)^m\partial_{x_i}$	$P^i_m = -t^{m+1}\partial_{x_i}$
$J = -x_1\partial_{x_2} + x_2\partial_{x_1}$	$J_m = -t^m(x_1\partial_{x_2} - x_2\partial_{x_1})$

one may show that

- L_m, J_m, P^i_m satisfy closed commutation relations
- those commutation relations satisfy the Jacobi identity

Infinite dimensional algebra

We thus have an ∞ dim Lie algebra

Martelli, Tachikawa 2010

$$L_m,\ J_m,\ P^i_m\ (m \in \mathbb{Z},\ i,j = 1,2)$$

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [J_m, J_n] = [P^i_m, P^j_n] = 0,$$

$$[L_m, J_n] = -nJ_{m+n}, \qquad [L_m, P^i_n] = (m-n)P^i_{m+n},$$

$$[J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}. \qquad \varepsilon_{12} = -\varepsilon_{21} = 1$$

$$(\ \langle L_m \rangle \ni\!\!\!\!+ \langle J_m \rangle\ ) \ni\!\!\!\!+ \langle P^i_m \rangle$$

(centerless Virasoro $\ni\!\!\!\!+$ Abelian) $\ni\!\!\!\!+$ Abelian ideal

$\langle J_m \rangle$, $\langle P^i_m \rangle$: adjoint representation of $\langle L_m \rangle$

Infinite dimensional algebra

central extension

Theorem (NA 2011)

The ∞ dim algebra has the following central extensions

- $[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12}m(m^2-1)\delta_{m+n,0},$
- $[J_m, J_n] = \beta m \delta_{m+n,0},$ α, β : central charge

$(\because)$ Jacobi identity

remarks

- Virasoro central extension
- $\langle J_m \rangle$ becomes non-Abelian, but $\langle P_m^i \rangle$ remains Abelian
- exotic central extension for the finite dim algebra does not survive:

$$[P_m^i, P_n^j] = I_{mn}\varepsilon^{ij}\Theta$$

Infinite dimensional algebra

We investigate representations of the following ∞ dim algebra $\mathfrak{g}$

Definition

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0},$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad [P^i_m, P^j_n] = 0, \qquad [L_m, J_n] = -nJ_{m+n},$$

$$[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}.$$

$m, n \in \mathbb{Z},\ i, j = 1, 2,\ \varepsilon_{12} = -\varepsilon_{21} = 1,\ \alpha, \beta$: const

Outline

1. finite dim planar Galilean conformal algebra
   definition / central extension
2. infinite dim planar Galilean conformal algebra
   definition / central extension
3. highest weight representations (Verma modules)
   Verma modules / Kac determinant / irreducibility of VM
4. coadjoint representation
   coadjoint reps of infinite dim algebra and group / example of coadjoint orbit
5. summary

Highest weight representation (Verma modules)

goal

show irreducibility of Verma modules by computing Kac determinant

Verma modules

- idea similar to boson Fock space
  - boson alg = creation op + annihilation op + diagonal
  - $^{\exists}\,|0\rangle$ s.t. $a\,|0\rangle = 0,\ H\,|0\rangle = \lambda\,|0\rangle$
  - representation space $\simeq \{\ (a^{\dagger})^n\,|0\rangle\,,\ |\ n \in \mathbb{Z}_+\ \}$ $\leftarrow$ Verma module V^{λ}
- importance of Verma modules

Theorem (vague)

Any highest weight module over a semisimple Lie algebra is isomorphic to a quotient of Verma module.

Dixmier, *Enveloping algebras*, North-Holland 1977

Verma modules over the ∞ dim algebra $\mathfrak{g}$

Triangular decomposition

$$\underbrace{\langle\, L_{-n}, J_{-n}, P^i_{-n} \,\rangle}_{\mathfrak{g}^-} \;\oplus\; \underbrace{\langle\, L_0, J_0, P^i_0 \,\rangle}_{\mathfrak{g}^0} \;\oplus\; \underbrace{\langle\, L_n, J_n, P^i_n \,\rangle}_{\mathfrak{g}^+} \qquad n \in \mathbb{Z}_+$$

$$\Rightarrow\; [\mathfrak{g}^0, \mathfrak{g}^\pm] \subseteq \mathfrak{g}^\pm$$

Definition (highest weight vector $|0\rangle$)

$$\mathfrak{g}^+ |0\rangle = 0, \quad L_0 |0\rangle = h |0\rangle, \quad J_0 |0\rangle = \mu |0\rangle, \quad P^i_0 |0\rangle = \rho_i |0\rangle$$

Definition (Verma modules)

$$V^\chi = U(\mathfrak{g}^-) \otimes |0\rangle, \qquad \chi = (h, \mu, \rho_1, \rho_2, \alpha, \beta)$$

V^χ : specified by highest weights (h, μ, ρ_i) and central charges (α, β)

Verma modules over the ∞ dim algebra $\mathfrak{g}$

more precisely ... vectors in V^χ

Level (n)	vectors	#(vectors)
0	$\lvert 0\rangle$	1
1	$L_{-1}\lvert 0\rangle,\ J_{-1}\lvert 0\rangle,\ P^i_{-1}\lvert 0\rangle$	4
2	$L_{-2}\lvert 0\rangle,\ L^2_{-1}\lvert 0\rangle,\ L_{-1}J_{-1}\lvert 0\rangle,\dots$	14
3	$L_{-3}\lvert 0\rangle,\ L_{-2}L_{-1}\lvert 0\rangle,\ L_{-1}J_{-1}P^1_{-1}\lvert 0\rangle,\dots$	40

level = − (sum of indices)

- vectors in V^χ are classified by level
- dim $V^\chi = \infty$
- V^χ is a rep space of $\mathfrak{g}$

Irreducibility of Verma modules

Question : V^χ is reducible ?

Theorem

V^χ is irreducible if Kac determinant at any level does not vanish.

Kac, Raina, *Highest weight representations of infinite dimensional Lie algebras*, World Scientific, 1987

Kac determinant at level n

$|v_i\rangle, i = 1, 2, \cdots$: basis of level n subspace of V^χ

Definition

$$\text{Kac det at level } n = \det(\langle v_i | v_j \rangle)$$

where the scalar product is defined by

$$\langle 0|0\rangle = 1, \qquad (X|0\rangle, Y|0\rangle) = \langle 0|\,\omega(X)Y|0\rangle$$

with anti-automorphism ω of $\mathfrak{g}$

$$\omega(L_m) = L_{-m}, \qquad \omega(J_m) = J_{-m}, \qquad \omega(P^i_m) = P^i_{-m}$$

Irreducibility of Verma modules

explicit form of Kac determinant NA 2011

Proposition

Kac determinant at level n is given by

$$\text{numerical const} \times (\rho_1^2 + \rho_2^2)^{\frac{1}{2}\sum_{n_1,n_2}\sum_{A_1,A_2} s(A_1)s(A_2)(w(A_1)+w(A_2))}$$

- $s(A), w(A)$ quantities determined by a partition of integer A
- independent of h, μ, α, β

immediate consequence of Proposition

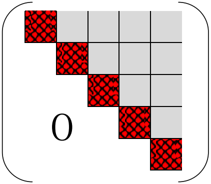
Theorem

V^χ *is irreducible if* $\rho_1^2 + \rho_2^2 \neq 0$

Irreducibility of Verma modules

Proof of the Kac determinant formula (sketch)

- introduction of an appropriate ordering of the vectors at level n
- Kac det $\sim$ det(matrix of row echelon form)



- Kac det $\sim$ $\prod$det(red square block)
- red square block $\sim$ specified by partitions of integer a, b ($n = a + b$)
- det(red square block) $\sim$ $\prod(\rho_1^2 + \rho_2^2)^{\frac{1}{2}s(A)s(B)(w(A)+w(B))}$ (induction)

for detail, NA arXiv:1112.0634 [math-ph]

Remarks on irreducibility of V^χ

Theorem

V^χ is irreducible if $\rho_1^2 + \rho_2^2 \neq 0$

- very different from Virasoro algebra
  – irreducibility depends on the central charge
- similar to Schrödinger-Virasoro algebra in $(1+1)$D spacetime

  Roger, Unterberger 2006
- intereting result or disappointing result ?
- $\rho_1^2 + \rho_2^2 = 0 \ \Rightarrow$ most probably V^χ is reducible
- interesting possibility ? $\cdots$ complex ρ_i satisfying $\rho_1^2 + \rho_2^2 = 0$,
  $e.g.\ \rho_1 = 1,\ \rho_2 = i$

Outline

1. finite dim planar Galilean conformal algebra
   definition / central extension
2. infinite dim planar Galilean conformal algebra
   definition / central extension
3. highest weight representations (Verma modules)
   Verma modules / Kac determinant / irreducibility of VM
4. coadjoint representation
   coadjoint reps of infinite dim algebra and group / example of coadjoint orbit
5. summary

Why coajoint representation ?

coadjoint representation of ∞ dim algebra $\mathfrak{g}$

$\downarrow$

coadjoint representation of ∞ dim group $\mathfrak{G}$

$\downarrow$

coajoint orbit of $\mathfrak{G}$ $\cdots$ symplectic manifold

open a way to physical applications

- Poisson bracket
  $\rightarrow$ classical dynamical system
- geometric quantization

Kirillov, *Lectures on the orbit method*, AMS 2004

work in progress...following the procedure of

C. Roger, J. Unterberger, *The Schrödinger-Virasoro Algebra: Mathematical structure and dynamical Schrödinger symmetries*, Springer, 2012

Coadjoint representation of ∞ dim algebra $\mathfrak{g}$

Lie algebra $\Rightarrow$ vector space

- rep of $\mathfrak{g}$ on $\mathfrak{g}$ $\cdots$ adjoint rep
- rep of $\mathfrak{g}$ on $\mathfrak{g}^*$ (vector space dual to $\mathfrak{g}$) $\cdots$ coadjoint rep of $\mathfrak{g}$

We thus have to find $\mathfrak{g}^*$ $\leftarrow$ difficult part

outline of finding $\mathfrak{g}^*$ Roger, Unterberger 2012

1. centerless Virasoro algebra $Vect(S^1)$
2. 1-parameter family of representation $\mathscr{F}_\lambda$ of $Vect(S^1)$
3. introduction of dual space $\mathscr{F}_\lambda^*$ to $\mathscr{F}_\lambda$
4. identification of $\mathfrak{g}$ (centerless) with sum of $\mathscr{F}_\lambda$
   (centerless $\mathfrak{g})^* \simeq$ sum of $\mathscr{F}_\lambda^*$
5. central extension

$\mathfrak{g}^*$: vector space dual to $\mathfrak{g}$

1. centerless Virasoro algebra $Vect(S^1)$
   $\Leftrightarrow$ Lie algebra of C^∞-vector fields on S^1 (θ : coordinate on S^1)

$$L_f := f(\theta)\partial_\theta \in Vect(S^1) \quad \Rightarrow \quad [L_f, L_g] = L_{fg'-f'g}$$

   Fourier components $\Rightarrow$ $[L_m, L_n] = (m-n)L_{m+n}$

2. rep of $Vect(S^1)$ on $\phi(\theta)d\theta^{-\lambda}$

$$L_f(\,\phi(\theta)d\theta^{-\lambda}\,) := (f\phi' - \lambda f'\phi)d\theta^{-\lambda}, \qquad -\lambda : \text{conformal weight}$$

   gives a 1-parameter family of rep of $Vect(S^1)$: $\mathscr{F}_\lambda$

   $(\because)$ direct computation shows that

$$(L_f L_g - L_g L_f)(\,\phi(\theta)d\theta^{-\lambda}\,) = L_{fg'-f'g}(\,\phi(\theta)d\theta^{-\lambda}\,)$$

$\mathfrak{g}^*$: vector space dual to $\mathfrak{g}$

3. dual space of $\mathscr{F}_\lambda$

   $\mathscr{F}_\lambda^* \simeq \mathscr{F}_{-\lambda-1}$ by the pairing $\mathscr{F}_\lambda^* \times \mathscr{F}_\lambda \to \mathbb{C}$

$$\langle\, u\, d\theta^{1+\lambda}, \phi\, d\theta^{-\lambda}\,\rangle = \int_{S^1} u(\theta)\phi(\theta) d\theta$$

4. the ∞ dim algebra (centerless)

   $\Leftrightarrow$ Lie algebra of vector fields on $S^1 \times \mathbb{R}^2$

$$L_f = f(\theta)\partial_\theta + f'(\theta) x_i \partial_{x_i}, \quad J_f = f(\theta)(x_1\partial_{x_2} - x_2\partial_{x_1}), \quad P_f^i = f(\theta)\partial_{x_i}$$

$$\Rightarrow\ [L_f, L_g] = L_{fg'-f'g}, \quad [L_f, P_g^i] = P^i_{fg'-f'g}, \quad [L_f, J_g] = J_{fg'}$$

$$[J_f, P_g^i] = -\sum_k \varepsilon_{ik} P^k_{fg}, \quad [J_f, J_g] = [P_f^i, P_g^j] = 0$$

Fourier components $\Rightarrow$ previously shown commutation relations

$\mathfrak{g}^*$: vector space dual to $\mathfrak{g}$

5. identification of the ∞ dim algebra (centerless) as $\mathscr{F}_\lambda$

com. relation	rep $\mathscr{F}_\lambda$ of $Vect(S^1)$
$[L_f, L_g] = L_{fg'-f'g}$	
$[L_f, P^i_g] = P^i_{fg'-f'g}$	$L_f(\ \phi(\theta)d\theta^{-\lambda}\ ) = (f\phi' - \lambda f'\phi)d\theta^{-\lambda}$
$[L_f, J_g] = J_{fg'}$	

commutation relation $\cdots$ adjoint action of L_f

one may make the identification

$$L_g \simeq \mathscr{F}_1, \quad P^i_g \simeq \mathscr{F}_1, \quad J_g \simeq \mathscr{F}_0$$

by $F^*_\lambda \simeq \mathscr{F}_{-\lambda-1}$

$$(\text{centerless algebra})^* \simeq \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-1}$$

$\mathfrak{g}^*$: vector space dual to $\mathfrak{g}$

6. central extension

$$\mathfrak{g} \simeq \text{ centerless algebra } \oplus \mathbb{R} \oplus \mathbb{R}$$

therefore

$$\mathfrak{g}^* \simeq \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-1} \oplus \mathbb{R} \oplus \mathbb{R}$$

duality pairing

$$\gamma_0 d\theta^2 + \gamma_1 d\theta^2 + \gamma_2 d\theta^2 + \gamma_3 d\theta + a + b \ \in \ \mathfrak{g}^*$$

$$\vec{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ a \\ b \end{pmatrix}, \quad \vec{X} = \begin{pmatrix} L_{f_0} \\ P^1_{f_1} \\ P^2_{f_2} \\ J_{f_3} \\ \alpha, \\ \beta \end{pmatrix}, \quad \langle \vec{\gamma}, \vec{X} \rangle = \sum_{i=0}^{3} \int_{S^1} \gamma_i f_i \, d\theta + a\alpha + b\beta$$

Coadjoint representation of ∞ dim algebra $\mathfrak{g}$

Proposition

Define an action $X(\gamma)$ of $\mathfrak{g}$ on $\mathfrak{g}^*$ by

$$\langle X(\gamma), Y \rangle := -\langle \gamma, [X, Y] \rangle, \quad X, Y \in \mathfrak{g}, \ \gamma \in \mathfrak{g}^*$$

then $X(\gamma)$ gives a representation of $\mathfrak{g}$.

Kirillov, *Lectures on the orbit method*, AMS 2004

$X(\gamma)$: dual of the adjoint representation $[X, Y]$

Coadjoint representation of ∞ dim algebra $\mathfrak{g}$

Proposition

Coadjoint representation of $\mathfrak{g}$ is given by

$$L_{f_0}\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix} af_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 \\ 2\gamma_1 f_0' + \gamma_1' f_0 \\ 2\gamma_2 f_0' + \gamma_2' f_0 \\ \gamma_3 f_0' + \gamma_3' f_0 \end{pmatrix}, \quad J_{f_3}(\vec{\gamma}) = \begin{pmatrix} \gamma_3 f_3' \\ \gamma_2 f_3 \\ -\gamma_1 f_3 \\ bf_3' \end{pmatrix},$$

$$P^1_{f_1}(\vec{\gamma}) = \begin{pmatrix} 2\gamma_1 f_1' + \gamma_1' f_1 \\ 0 \\ 0 \\ -\gamma_2 f_1 \end{pmatrix}, \qquad P^2_{f_2}(\vec{\gamma}) = \begin{pmatrix} 2\gamma_2 f_2' + \gamma_2' f_2 \\ 0 \\ 0 \\ -\gamma_1 f_2 \end{pmatrix}$$

action on the center a, b is trivial (omitted)

Coadjoint representation of ∞ dim algebra $\mathfrak{g}$

Proposition

Coadjoint representation of $\mathfrak{g}$ is given by

$$L_{f_0}\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix} af_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 \\ 2\gamma_1 f_0' + \gamma_1' f_0 \\ 2\gamma_2 f_0' + \gamma_2' f_0 \\ \gamma_3 f_0' + \gamma_3' f_0 \end{pmatrix}, \quad J_{f_3}(\vec{\gamma}) = \begin{pmatrix} \gamma_3 f_3' \\ \gamma_2 f_3 \\ -\gamma_1 f_3 \\ bf_3' \end{pmatrix},$$

$$P^1_{f_1}(\vec{\gamma}) = \begin{pmatrix} 2\gamma_1 f_1' + \gamma_1' f_1 \\ 0 \\ 0 \\ -\gamma_2 f_1 \end{pmatrix}, \qquad P^2_{f_2}(\vec{\gamma}) = \begin{pmatrix} 2\gamma_2 f_2' + \gamma_2' f_2 \\ 0 \\ 0 \\ -\gamma_1 f_2 \end{pmatrix}$$

action on the center a, b is trivial (omitted)

coadjoint representation of the Virasoro alg

Coadjoint orbit of ∞ dim group : an example

purpose

- a simple example of coadjoint orbit by coadjoint rep of $\mathfrak{g}$

full classification of orbits requires coadjoint rep of the ∞ dim group $\mathfrak{G}$

coadjoint orbit of $\mathfrak{G}$

- action of $\mathfrak{G}$ on $\mathfrak{g}^*$
- fixed $\gamma \in \mathfrak{g}^*$, $\quad \gamma \xrightarrow{\mathfrak{G}} \gamma' \;\Rightarrow\; \gamma'$ gives a manifold (coadjoint orbit)

how to find a coadjoint orbit ?

- find a subgroup $\mathfrak{H} \subseteq \mathfrak{G}$ s.t. γ is sable by $\mathfrak{H}$
- coadjoint orbit $\simeq \mathfrak{G}/\mathfrak{H}$

Coadjoint orbit of ∞ dim group : an example

generators of $\mathfrak{H}$

- by coadjoint rep of $\mathfrak{g}$,
- find infinitesimal transformations s.t. $\gamma \in \mathfrak{g}^*$ is stable
- namely, find $f_i(\theta)$ $(i = 0, 1, 2, 3)$ s.t.

$$(L_{f_0} + J_{f_3} - P^1_{f_1} - P^2_{f_2})(\vec{\gamma}) = 0$$

a simple case : γ_1, γ_2 nonvanishing constant

a set of differential equations

$$af_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 + \gamma_3 f_3' = 2(\gamma_1 f_1' + \gamma_2 f_2'),$$

$$2\gamma_1 f_0' + \gamma_2 f_3 = 0,$$

$$2\gamma_2 f_0' - \gamma_1 f_3 = 0,$$

$$\gamma_3 f_0' + \gamma_3' f_0 + bf_3' = -\gamma_2 f_1 + \gamma_1 f_2$$

Coadjoint orbit of ∞ dim group : an example

generators of $\mathfrak{H}$

- by coadjoint rep of $\mathfrak{g}$,
- find infinitesimal transformations s.t. $\gamma \in \mathfrak{g}^*$ is stable
- namely, find $f_i(\theta)$ $(i = 0, 1, 2, 3)$ s.t.

$$(L_{f_0} + J_{f_3} - P^1_{f_1} - P^2_{f_2})(\vec{\gamma}) = 0$$

a simple case : γ_1, γ_2 nonvanishing constant

a set of differential equations

$$af_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 + \gamma_3 f_3' = 2(\gamma_1 f_1' + \gamma_2 f_2'),$$

$$2\gamma_1 f_0' + \gamma_2 f_3 = 0, \rightarrow \text{eliminate } f_3 \ \rightarrow \ f_0 = const., \quad f_3 = 0$$

$$2\gamma_2 f_0' - \gamma_1 f_3 = 0, \nearrow \qquad \gamma_0' f_0 = 2(\gamma_1 f_1' + \gamma_2 f_2')$$

$$\gamma_3 f_0' + \gamma_3' f_0 + bf_3' = -\gamma_2 f_1 + \gamma_1 f_2 \qquad \gamma_3' f_0 = -\gamma_2 f_1 + \gamma_1 f_2$$

Coadjoint orbit of ∞ dim group : an example

solution

$$f_0 = const., \quad f_3 = 0, \quad f_k = \Phi_k(\theta) f_0 + const, \;\; k = 1, 2$$

$\Phi_k(\theta)$: functions of $\gamma_0(\theta)$, $\gamma_3(\theta)$

generator of $\mathfrak{H}$

$$L_{f_0} = cons.\partial_t, \qquad P^k_{f_k} = f_k(\theta)\partial_{x_k}, \qquad J_{f_3} = 0$$

$\downarrow$ translation on S^1 $\qquad$ $\downarrow$ no restriction on $f_k(\theta)$

$\therefore\ \mathfrak{H} \simeq U(1) \times \mathfrak{G}_P$(group generated by $P^k_{f_k}$)

coadjoint orbit (γ_1, γ_2 nonvanishing constant)

$Vir/S^1 \times \widehat{SO(2)}$ $\qquad$ Vir : Virasoro group

The ∞ dim group $\mathfrak{G}$

L_f : generate Virasoro group *Vir* (well-known)

subgroup generated by $J_\xi, \ P^1_{\eta_1}, \ P^2_{\eta_2}$

- $(\xi, \eta_1, \eta_2) \ \sim \ \exp(J_\xi)\exp(P^1_{\eta_1} + P^2_{\eta_2})$
- multiplication

$$(\xi, \eta_1, \eta_2)(\rho, \sigma_1, \sigma_2) = \\ (\xi + \rho, \sigma_1 + \eta_1 \cos\rho - \eta_2 \sin\rho, \sigma_2 + \eta_1 \sin\rho + \eta_2 \cos\rho)\exp(c\beta)$$

- inverse

$$(\xi, \eta_1, \eta_2)^{-1} = (-\xi, -\eta_1 \cos\xi - \eta_2 \sin\xi, \eta_1 \sin\xi - \eta_2 \cos\xi)$$

whole group $\mathfrak{G}$

$$(\phi, \xi, \eta_1, \eta_2) := (id, \xi, \eta_1, \eta_2)(\phi, 0, 0, 0), \qquad \phi \in Vir$$

Coadjoint representation of $\mathfrak{G}$

definition

$X \in \mathfrak{g}, \quad \gamma \in \mathfrak{g}^*, \quad g \in \mathfrak{G}$

Proposition

Define an action $g(\gamma)$ of $\mathfrak{G}$ on $\mathfrak{g}^*$ by

$$\langle g(\gamma), X \rangle := \langle \gamma, g^{-1} X g \rangle$$

then $g(\gamma)$ gives a representation of $\mathfrak{G}$.

Kirillov, *Lectures on the orbit method*, AMS 2004

recall that

$\mathfrak{g}^*$ is a representation space of Virasoro alg

$$\mathfrak{g}^* \simeq (\text{Virasoro alg})^* \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-1} \oplus \mathbb{R}$$

$\Rightarrow$ action of *Vir* on $\mathfrak{g}^*$ is easily computed

Coadjoint representation of $\mathfrak{G}$

Proposition

Coadjoint representation of *Vir* subgroup is given by

$$\phi(\gamma) = \begin{pmatrix} a\Theta(\phi) + (\gamma_0 \circ \phi)(\phi')^2 \\ (\gamma_1 \circ \phi)(\phi')^2 \\ (\gamma_2 \circ \phi)(\phi')^2 \\ (\gamma_3 \circ \phi)\phi' \end{pmatrix}, \quad \Theta(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$$

coadjoint representation of *Vir* $\qquad$ $\Theta(\phi)$: Schwarzian derivative

$(\because)$ $\mathfrak{g}^* \simeq (\text{Virasoro alg})^* \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-1} \oplus \mathbb{R}$

coadjoint rep of *Vir* $\cdots$ well-known formula

coadjoint action of *Vir* on other $\mathscr{F}_{-\lambda}$

$$Vir : \theta \mapsto \theta = \phi(t), \quad \gamma(\theta) d\theta^\lambda = \gamma(\phi(t)) \left(\frac{d\theta}{dt} dt\right)^\lambda = (\gamma \circ \phi)(\phi')^\lambda dt^\lambda$$

Coadjoint representation of $\mathfrak{G}$

Proposition

Coadjoint representation of (ξ, η_1, η_2) subgroup is given by

$$(\xi, \eta_1, \eta_2)(\gamma) = \begin{pmatrix} \gamma_0 + \sum_{k=1,2}(\gamma_k' \eta_k + 2\gamma_k \eta_k') + (\gamma_1\eta_2 - \gamma_2\eta_1 + \gamma_3)\xi' + \frac{b}{2}(\xi')^3 \\ \gamma_1 \cos\xi + \gamma_2 \sin\xi \\ -\gamma_1 \sin\xi + \gamma_2\cos\xi \\ \gamma_1\beta_2 - \gamma_2\beta_1 + \gamma_3 + b\xi' \end{pmatrix}$$

(∵) direct computation according to the definition. *e.g.*

$$g = (\xi, \eta_1, \eta_2) \ \Rightarrow\ g^{-1} P^1_{f_1} g = P^1_{f_1} \cos\xi + P^2_{f_2} \sin\xi,$$

$$\langle g(\gamma), P^1_{f_1}\rangle := \langle \gamma, g^{-1} P^1_{f_1} g\rangle = \int_{S^1} (\gamma_1 \cos\xi + \gamma_2 \sin\xi) f_1 \, d\theta$$

Outlook

on coadjoint representation of $\mathfrak{G}$

- classification of orbits ... may be done
- geometric quantization

on action of $\mathfrak{G}$ on differential operators

- Virasoro group $\cdots$ Hill op: $L_u = \partial_z^2 + u(z), \ z \in \mathbb{C}, \ |z| = 1$
  Lax form of KdV eq. : $\partial_t L_u = 3[P_u, L_u]$

$$Vir: \ z \mapsto \phi(z), \quad L_u \mapsto (\phi')^{-3/2} L_{\tilde{u}} (\phi')^{-1/2}, \quad \tilde{u} = (u \circ \phi)(\phi')^2 + \frac{1}{2}\Theta(\phi)$$

- Schrödinger-Virasoro group in $(1+1)D$ $\cdots$ Schrödinger op

$$-2im\partial_t - \partial_x^2 + u(t,x) \quad \mapsto \quad -2im\partial_t - \partial_x^2 + \tilde{u}(t,x)$$

- Galilean conformal group in $(2+1)$D ?

Summary

We have studied mathematical aspects of ∞ dim planar Galilean conformal algebra by Martelli and Tachikawa.

1. highest weight representation
   $\rho_1^2 + \rho_2^2 \neq 0 \;\Rightarrow\;$ Verma modules are irreducible
2. coadjoint representations
   - derivation of coadjoint reps of the algebra $\mathfrak{g}$ and the group $\mathfrak{G}$
   - coadjoint orbit for a simple case

other ∞ dim extensions

A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, arXiv:0912.1090 [hep-th]

A. Hosseiny and S. Rouhani, arXiv:1001.1036 [hep-th]

M. Henkel, R. Schott, S. Stoimenov and J. Unterberger, arXiv:math-ph/0601028.

A. Bagchi, Gopakumar, JHEP **07** (2009) 037; arXiv:0902.1385 [hep-th]

M. Alishahiha, A. Davody and A. Vahedi, JHEP **08** (2009) 022; arXiv:0903.3953 [hep-th]

A. Hosseiny and S. Rouhani, J. Math. Phys. **51** (2010) 052307; arXiv:0909.1203 [hep-th]

Additional slides

Current algebra with central extension

$$[L_f, L_g] = L_{fg'-f'g} - \frac{i}{48\pi}\int_0^{2\pi}(fg''' - f'''g)d\theta,$$

$$[J_f, J_g] = \frac{i}{4\pi}\int_0^{2\pi}(fg' - f'g)d\theta,$$

$$[L_f, P^i_g] = P^i_{fg'-f'g}, \quad [L_f, J_g] = J_{fg'},$$

$$[J_f, P^i_g] = -\sum_k \varepsilon_{ik} P^k_{fg}, \quad [P^i_f, P^j_g] = 0$$

Review of Galilean conformal algebras in $(d+1)$D spacetime

Spin "ℓ" Galilean conformal algebras Negro *et al.* 1997

Infinitesimal transformations in $(d+1)$ dim spacetime

$\ell \in \frac{1}{2}\mathbb{Z}_+ \quad i,j = 1,\dots,d$

- time translation $H = \frac{\partial}{\partial t}$ $\quad t \to t + \varepsilon$
- rotation $J_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}$ $\quad x_i \to x_i + \varepsilon x_j, \quad x_j \to x_j - \varepsilon x_i$
- $P_i^n = (-t)^n \frac{\partial}{\partial x_i}, \quad n = 0,\dots,2\ell$
  - $n = 0$ (translation) $\quad x_i \to x_i + \varepsilon$
  - $n = 1$ (Galilean boost) $\quad x_i \to x_i - \varepsilon t$
  - $n = 2$ (acceleration) $\quad x_i \to x_i + \varepsilon t^2$
- dilatation $D = -t\frac{\partial}{\partial t} - \ell x_i \frac{\partial}{\partial x_i}$ $\quad t \to (1-\varepsilon)t, \ x_i \to (1-\ell\varepsilon)x_i$
- conformal transformation $C = t^2 \frac{\partial}{\partial t} + 2\ell t x_i \frac{\partial}{\partial x_i}$

  $t \to (1+\varepsilon t)t, \ x_i \to (1+2\ell\varepsilon t)x_i$

Spin "ℓ" Galilean conformal algebras Negro *et al.* 1997

Lie algebra structure

$$(\langle D, H, C \rangle \oplus \langle J_{ij} \rangle) \ni\!\!\!\!+ \langle P_i^n \rangle$$

$$i,j=1,\cdots,d, \quad n=0,\ldots,2\ell, \quad \ell \in \tfrac{1}{2}\mathbb{Z}_+$$

$$(sl(2,\mathbb{R}) \quad \oplus \quad so(n)) \quad \ni\!\!\!\!+ \quad \text{Abelian ideal}$$

e.g. $\ell = \frac{1}{2} \Rightarrow$ Schrödinger algebra

Spin "ℓ" Galilean conformal algebras Negro *et al.* 1997

Lie algebra structure

$$(\langle D,\, H,\, C \rangle \oplus \langle J_{ij} \rangle) \niplus \langle P_i^n \rangle$$

$$i,j{=}1,\cdots,d, \quad n{=}0,\ldots,2\ell, \quad \ell \in \tfrac{1}{2}\mathbb{Z}_+$$

$$(sl(2,\mathbb{R}) \quad \oplus \quad so(n)) \quad \niplus \quad \text{Abelian ideal}$$

e.g. $\ell = \frac{1}{2} \Rightarrow$ Schrödinger algebra

Infinite dimensional extension Martelli, Tachikawa 2010

$$L^m = -t^{m+1}\partial_t - \ell(m+1)t^m \sum_{i=1}^{d} x_i \partial_i,$$

$$J_{ij}^m = -t^m(x_i\partial_j - x_j\partial_i), \qquad P_i^n = -t^{n+\ell}\partial_i.$$

$$i,j{=}1,\cdots,d, \quad m \in \mathbb{Z}, \quad n \in \mathbb{Z}+\ell$$

Spin "ℓ" Galilean conformal algebras Negro *et al.* 1997

Lie algebra structure

$$(\, \langle\, D,\, H,\, C\,\rangle \,\oplus\, \langle\, J_{ij}\,\rangle\,) \,\niplus\, \langle\, P^n_i\,\rangle$$

$$i,j{=}1,\cdots,d, \quad n{=}0,\ldots,2\ell, \quad \ell \in \tfrac{1}{2}\mathbb{Z}_+$$

$$(\, sl(2,\mathbb{R}) \quad \oplus \quad so(n)\,) \quad \niplus \quad \text{Abelian ideal}$$

e.g. $\ell = \frac{1}{2} \;\Rightarrow$ Schrödinger algebra

Infinite dimensional extension Martelli, Tachikawa 2010

$$(\,\langle L^m\,\rangle \,\niplus\, \langle\, J^m_{ij}\,\rangle\,) \,\niplus\, \langle\, P^n_i\,\rangle \qquad i,j{=}1,\cdots,d, \quad m\in\mathbb{Z},\; n\in\mathbb{Z}+\ell, \quad \ell\in\tfrac{1}{2}\mathbb{Z}$$

$$(\,\text{Virasoro} \niplus \widehat{so(n)}\,) \niplus \text{Abelian ideal}$$

e.g. $\ell = \frac{1}{2} \;\Rightarrow$ Schrödinger-Virasoro algebra Henkel 1994

Central extensions

Finite dim algebra
Martelli, Tachikawa 2010

two types of extensions

- any d and half-integral ℓ

$$[P_i^m, P_j^n] = I^{mn}\delta_{ij} M$$

- only $d = 2$ and integral ℓ $\Rightarrow$ *exotic* extension

$$[P_i^m, P_j^n] = I^{mn}\varepsilon_{ij}\Theta, \quad \varepsilon_{12} = -\varepsilon_{21} = 1$$

Infinite dim algebra

Not studied yet !

$d=2,\ \ell=1$ Infinite dimensional algebra

- classification of central extension
- boson realization
- highest weight modules (Verma modules)

Relations and structure

$L_m,\ J_m,\ P^i_m\ (m \in \mathbb{Z},\ i,j = 1,2)$ (slight change of notation)

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [J_m, J_n] = [P^i_m, P^j_n] = 0,$$

$$[L_m, J_n] = -nJ_{m+n}, \qquad [L_m, P^i_n] = (m-n)P^i_{m+n},$$

$$[J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}. \qquad \varepsilon_{12} = -\varepsilon_{21} = 1$$

$$(\ \langle L_m \rangle \ni\!\!\!\!+ \langle J_m \rangle\ ) \ni\!\!\!\!+ \langle P^i_m \rangle$$

(Virasoro $\ni\!\!\!\!+$ Abelian) $\ni\!\!\!\!+$ Abelian ideal

Relations and structure

$L_m,\ J_m,\ P^i_m\ (m \in \mathbb{Z},\ i,j = 1,2)$ (slight change of notation)

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [J_m, J_n] = [P^i_m, P^j_n] = 0,$$

$$[L_m, J_n] = -nJ_{m+n}, \qquad [L_m, P^i_n] = (m-n)P^i_{m+n},$$

$$[J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}. \qquad \varepsilon_{12} = -\varepsilon_{21} = 1$$

$$(\ \langle L_m \rangle \ni\!\!\!\!+ \langle J_m \rangle\ ) \ni\!\!\!\!+ \langle P^i_m \rangle$$

(Virasoro $\ni\!\!\!\!+$ Abelian) $\ni\!\!\!\!+$ Abelian ideal

Q : Does the algebra have exotic type central extension ? $[P^i_m, P^j_n] \sim \varepsilon_{ij}$

Relations and structure

$L_m,\ J_m,\ P^i_m\ (m \in \mathbb{Z},\ i,j = 1,2)$ (slight change of notation)

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [J_m, J_n] = [P^i_m, P^j_n] = 0,$$

$$[L_m, J_n] = -nJ_{m+n}, \qquad [L_m, P^i_n] = (m-n)P^i_{m+n},$$

$$[J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}. \qquad \varepsilon_{12} = -\varepsilon_{21} = 1$$

$$(\ \langle L_m \rangle \ni\!\!\!\!+ \langle J_m \rangle\ ) \ni\!\!\!\!+ \langle P^i_m \rangle$$

(Virasoro $\ni\!\!\!\!+$ Abelian) $\ni\!\!\!\!+$ Abelian ideal

Q : Does the algebra have exotic type central extension ? $[P^i_m, P^j_n] \sim \varepsilon_{ij}$

A : No !

Central extensions

Theorem

All possible central extensions are listed as follows:

- $[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0}$,
- $[J_m, J_n] = \beta m \delta_{m+n,0}$, $\qquad \alpha, \beta$: *const*

(Proof) Adding central terms to each commutators

$$[L_m, L_n] = (m-n)L_{m+n} + Z^{(L)}_{mn}, \qquad [J_m, J_n] = Z^{(J)}_{mn}, \qquad [P^i_m, P^j_n] = Y^{ij}_{mn},$$

$$[L_m, J_n] = -nJ_{m+n} + C_{mn}, \qquad [L_m, P^i_n] = (m-n)P^i_{m+n} + F_{mn}^{\ \ i},$$

$$[J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n} + W_{mn}^{\ \ i}.$$

check the Jacobi identity and whether the extensions are absorbed in redefinition of generators.

Central extensions

Theorem

All possible central extensions are listed as follows:

- $[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0},$
- $[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad \alpha, \beta : const$

(Proof) relations from the Jacobi identity

$$(k-m)Z^{(L)}_{k+m\ n} + (m-n)Z^{(L)}_{m+n\ k} + (n-k)Z^{(L)}_{n+k\ m} = 0, \quad kZ^{(J)}_{k+m\ n} - nZ^{(J)}_{n+m\ k} = 0,$$
$$(m-k)C_{m+k\ n} - nC_{k+n\ m} + nC_{n+m\ k} = 0, \qquad Y^{11}_{m+k\ n} + Y^{22}_{m+n\ k} = 0,$$
$$(m-k)Y^{ij}_{m+k\ n} - (m-n)Y^{ji}_{m+n\ k} = 0, \qquad Y^{ij}_{m+k\ n} - Y^{ij}_{m+n\ k} = 0, \quad i \neq j,$$
$$(m-n)F_{m+n\ k}^{\ i} - (n-k)F_{m\ n+k}^{\ i} + (m-k)F_{n\ m+k}^{\ i} = 0,$$
$$F_{m\ n+k}^{\ i} = \sum_j \varepsilon_{ij}(nW_{m+n\ k}^{\ j} - (m-k)W_{n\ m+k}^{\ j}), \qquad W_{m\ n+k}^{\ i} - W_{n\ m+k}^{\ i} = 0.$$

One may verify that $Y^{ij}_{mn} = 0$ and $F^i_{mn},\ W^i_{mn}$ are trivial

Representations

Investigate the representations of the algebra with central extensions

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0},$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad [P^i_m, P^j_n] = 0, \qquad [L_m, J_n] = -nJ_{m+n},$$

$$[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}.$$

$m, n \in \mathbb{Z},\ i,j = 1,2,\ \varepsilon_{12} = -\varepsilon_{21} = 1,\ \alpha, \beta$: const

Boson realization

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0},$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad [P^i_m, P^j_n] = 0, \qquad [L_m, J_n] = -nJ_{m+n},$$

$$[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}.$$

$m, n \in \mathbb{Z},\ i, j = 1, 2,\ \varepsilon_{12} = -\varepsilon_{21} = 1,\ \alpha, \beta$: const

1. L_m, J_m sector $(\alpha = \beta = 1)$ $\Rightarrow$ boson realization of Virasoro alg.

Wakimoto, Yamada 1983

$$[a_m, a_n] = m\delta_{m+n,0}, \quad L_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} : a_k a_{m-k} :, \quad J_m = a_m \quad m, n \in \mathbb{Z}$$

Boson realization

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12}m(m^2-1)\delta_{m+n,0},$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad [P^i_m, P^j_n] = 0, \qquad [L_m, J_n] = -nJ_{m+n},$$

$$[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}.$$

$m, n \in \mathbb{Z},\ i,j = 1,2,\ \varepsilon_{12} = -\varepsilon_{21} = 1,\ \alpha, \beta$: const

2. three more independent bosons

$$[b_i, \bar{b}_j] = \delta_{i,j}, \quad [b_i, b_j] = [\bar{b}_i, \bar{b}_j] = 0, \quad [c, \bar{c}] = 1. \qquad i = 1,2$$

Boson realization

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0},$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \qquad [P^i_m, P^j_n] = 0, \qquad [L_m, J_n] = -nJ_{m+n},$$

$$[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [J_m, P^i_n] = \sum_j \varepsilon_{ij} P^j_{m+n}.$$

$m, n \in \mathbb{Z},\ i,j = 1,2,\ \varepsilon_{12} = -\varepsilon_{21} = 1,\ \alpha, \beta$: const

3. whole algebra ($\alpha = \beta = 1$) is realized by

$$L_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} : a_k a_{m-k} : -\bar{c}^{m+1} c - (m+1)c^m \sum_i \bar{b}_i b_i,$$

$$J_m = a_m - \bar{c}^m \sum_{ij} \varepsilon_{ij} \bar{b}_i b_j, \qquad P^i_m = -\bar{c}^{m+1} b_i.$$

$d = 2,\ \ell = 1$ Finite dimensional algebra

Definition

Generators

D	:	dilatation	C	:	conformal	H	:	time translation
J	:	rotation	P_i^0	:	space trans.	P_i^1	:	Galilei boost
P_i^2	:	acceleration			i = 1, 2			

Nonvanishing commutators

$$[D,H]=H, \qquad [D,C]=-C, \qquad [C,H]=2D,$$
$$[H,P_i^n]=-nP_i^{n-1} \quad [D,P_i^n]=(1-n)P_i^n, \quad [C,P_i^n]=(2-n)P_i^{n+1},$$
$$[J,P_i^n]=\sum_j \varepsilon_{ij}\delta_{ik}P_j^n, \qquad \varepsilon_{12}=-\varepsilon_{21}=1$$

Exotic central extension

$$[P_i^1,P_j^1]=\Theta\varepsilon_{ij}, \quad [P_i^0,P_j^2]=-2\Theta\varepsilon_{ij}$$

Highest weight representation (Verma modules) NA, Isaac 2011

Change of basis

$$P_\pm = P_1^0 \pm 1 P_2^0, \quad K_\pm = P_1^1 \pm i P_2^1, \quad F_\pm = P_1^2 \pm i P_2^2, \quad J = iJ, \quad \Theta = i\Theta$$

Triangular decomposition

$$\mathfrak{g}^0 = \langle\ D,\ J,\ \Theta\ \rangle, \quad \mathfrak{g}^+ = \langle\ H,\ P_\pm,\ K_+\ \rangle, \quad \mathfrak{g}^- = \langle\ C,\ F_\pm,\ K_-\ \rangle,$$

$$[\mathfrak{g}^0, \mathfrak{g}^\pm] \subset \mathfrak{g}^\pm, \qquad [\mathfrak{g}^+, \mathfrak{g}^-] \subset \mathfrak{g}^0$$

Highest weight vector $|d, r\rangle$

$$D\,|d,r\rangle = d\,|d,r\rangle, \quad J\,|d,r\rangle = r\,|d,r\rangle, \quad \Theta\,|d,r\rangle = \theta\,|d,r\rangle, \quad \mathfrak{g}^+\,|d,r\rangle = 0.$$

Verma module $V^{d,r}$

$$V^{d,r} = U(\mathfrak{g}^-)\,|d,r\rangle = \{\ C^h K_-^k F_-^\ell F_+^m\,|d,r\rangle\ \}$$

Highest weight representation (Verma modules)

Theorem

$V^{d,r}$ is irreducible $\Leftrightarrow$ $V^{d,r}$ has no singular vectors

singular vector $|s\rangle$ $\cdots$ another h.w. vector in $V^{d,r}$

$$|s\rangle \neq \mathbb{C}|d,r\rangle\,, \quad D\,|s\rangle = d'\,|s\rangle\,, \quad J\,|s\rangle = r'\,|s\rangle\,, \quad \Theta\,|s\rangle = \theta\,|s\rangle\,,$$

$$H\,|s\rangle = P_{\pm}\,|s\rangle = K_{+}\,|s\rangle = 0.$$

Theorem

$V^{d,r}$ has precisely one singular vector iff $2d + 3 \in \mathbb{Z}_+$

$$|s\rangle = (2\Theta C - K_- F_+)^{2d+3}\,|d,r\rangle$$

Irreducibility of Verma modules

$2d+3 \in \mathbb{Z}_+ \ \Rightarrow\ V^{d,r}$ reducible

Search singular vectors in $\tilde{V}^{d,r} = V^{d,r}/\mathscr{I}, \qquad \mathscr{I} = U(\mathfrak{g}^-)|s\rangle$
$\Rightarrow$ no singular vectors in $\tilde{V}^{d,r}$

Theorem

All irreducible H.W. modules over the finite dim GCA are listed as follows

- $V^{d,r}$ *for* $2d+3 \notin \mathbb{Z}_+$
- $\tilde{V}^{d,r}$ *for* $2d+3 \in \mathbb{Z}_+$

All modules are ∞ dimensional.

Invariant equations

Definition

G : Lie group

$T : (x_1, x_2, \cdots, x_n) \to (x_1', x_2', \cdots, x_n')$ transformation generated by G

partial diff eq $\hat{S}\psi(x_1, x_2, \cdots, x_n) = 0$ is G-invariant if

$$\hat{S}\psi(x_1, x_2, \cdots, x_n) = 0 \quad \Rightarrow \quad \hat{S}(T\psi(x_1, x_2, \cdots, x_n)) = 0$$

Canonical method for semisimple G Kostant 1975, Dobrev 1988

- vector field rep of $\mathfrak{g} = \mathrm{Lie}(G)$ on G
- substitution into the singular vectors $\to$ invariant eq

$$g = \exp(tC + x_+ F_+ + x_- F_- + kK_-) \in \exp(\mathfrak{g}^-) \qquad \text{4 dim manifold}$$

$$T \in \text{vector field rep} : Tf(g) = \frac{d}{d\xi} f(e^{-\xi T} g)\Big|_{\xi=0}$$

Apply this for the non-semisimple GCA

Invariant equations

Vector filed representation

$$C = -\partial_t - \frac{k}{2}\partial_{x_-}, \qquad F_\pm = -\partial_{x_\pm}, \qquad K_- = -\partial_k + \frac{t}{2}\partial_{x_-}$$

$$\hat{D} = -d, \qquad \hat{J} = -r, \qquad \hat{\Theta} = -\theta$$

Theorem

The hierarchy of equations below is invariant under GCA

$$\left(2\theta\partial_t + \frac{\partial^2}{\partial x_+ \partial k} - t\frac{\partial^2}{\partial x_+ \partial x_-}\right)^{2d+3} \psi(t, x_\pm, k) = 0$$

Lowest member of hierarchy $(2d+3=1)$

$$\left(2\theta\partial_t + \frac{\partial^2}{\partial x_+ \partial k} - t\frac{\partial^2}{\partial x_+ \partial x_-}\right)\psi(t,x_\pm,k) = 0$$

Separation of variables $\psi(t,x_\pm,k) = \phi(t,x_\pm)f(k)$

$$\left(\partial_t - \frac{t}{2\theta}\frac{\partial^2}{\partial x_+ \partial x_-} + \frac{\lambda}{2\theta}\partial_{x_+}\right)\phi(t,x_\pm) = 0$$

$$\frac{df(k)}{dk} = \lambda f(k)$$

change of variable $x_+ \ \rightarrow \ x_+ - \frac{\lambda}{2\theta}t$

$$\left(\partial_t - \frac{t}{2\theta}\frac{\partial^2}{\partial x_+ \partial x_-}\right)\phi(t,x_\pm) = 0$$

free Schrödinger equation with time-dependent mass

Nonvanishing commutators

$\ell \in \mathbb{Z} + \frac{1}{2}, \quad i,j = 1,\ldots,d$

Finite dim algebra

$$[D,H] = H, \qquad [D,C] = -C, \qquad [C,H] = 2D,$$
$$[H,P_i^n] = -nP_i^{n-1} \quad [D,P_i^n] = (\ell - n)P_i^n, \quad [C,P_i^n] = (2\ell - n)P_i^{n+1},$$
$$[J_{ij},P_k^n] = \delta_{ik}P_j^n - \delta_{jk}P_i^n,$$
$$[J_{ij},J_{k\ell}] = \delta_{ik}J_{j\ell} + \delta_{j\ell}J_{ik} - \delta_{i\ell}J_{jk} - \delta_{jk}J_{i\ell}. \qquad n = 0,\ldots,2\ell$$

Infinite dim algebra

$$[L^m,L^n] = (m-n)L^{m+n}, \qquad [L^m,J_{ij}^n] = -nJ_{ij}^{n+m},$$
$$[J_{ij}^m,J_{k\ell}^n] = \delta_{i,k}J_{j\ell}^{m+n} + \delta_{j,\ell}J_{ik}^{m+n} - \delta_{i,\ell}J_{jk}^{m+n} - \delta_{jk}J_{i\ell}^{m+n},$$
$$[L^m,P_i^n] = (\ell m - n)P_i^{m+n}, \qquad [J_{ij}^m,P_k^n] = \delta_{i,k}P_j^{m+n} - \delta_{j,k}P_i^{m+n},$$

$n \in \mathbb{Z}$ for L^n, J_{ij}^n and $n \in \mathbb{Z} + \ell$ for P_i^n.

Vector field representation

Finite dim algebra

$$H = \frac{\partial}{\partial t}, \; D = -t\frac{\partial}{\partial t} - l x_i \frac{\partial}{\partial x_i}, \; C = t^2 \frac{\partial}{\partial t} + 2ltx_i \frac{\partial}{\partial x_i},$$

$$J_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \; P_i^n = (-t)^n \frac{\partial}{\partial x_i},$$

infinite dim algebra

$$L^m = -t^{m+1}\partial_t - \ell(m+1)t^m \sum_{i=1}^{d} x_i \partial_i,$$

$$J_{ij}^m = -t^m(x_i\partial_j - x_j\partial_i), \qquad P_i^n = -t^{n+\ell}\partial_i.$$

Relation to finite dim algebra

$$L^{-1} = -H, \quad L^0 = D, \quad L^1 = -C, \quad J_{ij}^0 = J_{ij},$$

$$P_i^{-\ell}, \; P_i^{-\ell+1}, \; \cdots, \; P_i^{\ell} \text{ (with multiplication of } \pm 1)$$