Lie symmetries of (2+1)-dimensional nonlinear Dirac equations

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Abstract

A preliminary study of Lie symmetries in a class of nonlinear Dirac equations in two spatial dimensions is given. A list of nonlinearities for which such equations admit one-dimensional extension of the maximal Lie invariance algebras is presented. The obtained symmetries can be used for construction of closed-form solutions for the equations under study.

1. Introduction

In 2004/2005 the first truly two-dimensional solid state material, graphene, was created in the laboratory [2, 3]. Later in 2010 the Nobel Prize in Physics was awarded to A. Geim and K. Novoselov "for groundbreaking experiments regarding the two-dimensional material graphen". It was noted in [2] that in graphene "electron transport is essentially governed by Dirac's (relativistic) equation." The authors of [3] have presented a new class of nonlinear phenomena in Bose–Einstein condensates in a honeycomb optical lattice, that can be described by a nonlinear Dirac equation (NLDE) in (2+1)-dimensions. The form of the nonlinearity appeared as a natural physical result of binary interactions between bosons. It was shown that NLDE for Bose–Einstein condensates breaks the Lie symmetry governed by Poincaré algebra. After these works there were a number of papers with studies of NLDEs in two spatial dimensions. For example, some exact stationary state solutions of a nonlinear Dirac equation in 2+1 dimensions was constructed in [5].

It is known that Lie symmetries present by themselves a powerful tool for construction of exact solutions of differential equations [6, 7]. That is why a number of physical models were studied with the Lie symmetry point of view (see important results, including those on NLDEs, in [8, 9]).

Inspired by the importance of (2+1)-dimensional NLDEs we investigate Lie symmetries of the following model equations

$$(i\sigma_2\partial_t + \sigma_1\partial_x - \sigma_3\partial_y)\Psi = \Phi, \qquad (1)$$

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where $\Psi = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\Phi = \begin{pmatrix} F \\ G \end{pmatrix}$. Here $u = u(t,x,y)$ and $v = v(t,x,y)$ are dependent variables, $F = F(u,v)$ and $G = G(u,v)$ are arbitrary smooth functions which are not linear in u and v simultaneously, σ_1, σ_2, and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equation (1) can be rewritten as the coupled system of first-order partial differential equations (PDE system)

$$\begin{aligned} v_t + v_x - u_y &= F(u,v), \\ u_t - u_x - v_y &= -G(u,v). \end{aligned} \tag{2}$$

It is well known that the Dirac equation describes a complex wave function. However, for some physical problems it is possible to restrict ourselves to real solutions like it is done in theories of massive neutrino. In this case a special (Majorana) representation of the Dirac matrices with purely imaginary entries should be used.

In the present paper we also restrict ourselves to the Dirac equation for real wave functions. A more complicated classification problem for the complex nonlinear Dirac equation will be a subject of our further research.

2. Lie symmetries

2.1. Determining equations

We study Lie symmetries of PDE systems from class (2) using the classical approach [6, 7]. We fix a system $\mathcal{L}$ from class (2) and search for vector fields of the form

$$\begin{aligned} X = {} & \xi^t(t,x,y,u,v)\partial_t + \xi^x(t,x,y,u,v)\partial_x + \xi^y(t,x,y,u,v)\partial_y \\ & + \eta^u(t,x,y,u,v)\partial_u + \eta^v(t,x,y,u,v)\partial_v \end{aligned}$$

that generate one-parameter point symmetry groups of $\mathcal{L}$. These vector fields form the maximal Lie invariance algebra $A^{\max} = A^{\max}(\mathcal{L})$ of the PDE system $\mathcal{L}$. Any such vector field X satisfies the infinitesimal invariance criterion, i.e., the action of the first prolongation $X^{(1)}$ of X on $\mathcal{L}$ results in the condition identically satisfied for all solutions of this system. Namely, we require that

$$\begin{aligned} X^{(1)}(v_t + v_x - u_y - F(u,v))|_{\mathcal{L}} &= 0, \\ X^{(1)}(u_t - u_x - v_y + G(u,v))|_{\mathcal{L}} &= 0. \end{aligned} \tag{3}$$

After the elimination of u_t and v_t by means of (2), equations (3) become identities in nine variables, t, x, y, u, v, u_x, u_y, v_x and v_y. In fact, equations (3) are polynomials in the variables u_x, u_y, v_x and v_y. The coefficients of different powers of these variables are zero, which gives twenty four determining equations for the coefficients ξ^t, ξ^x, ξ^y, η^u and η^v.[1] At first we solve those determining equations, which do not involve arbitrary elements F and G, and find that the coefficients of the operator X have the form

$$\begin{aligned}
\xi^t &= \tfrac{1}{2}\alpha_0(t^2+x^2+y^2)+(\delta+\alpha_1 x+\alpha_2 y)t+\beta_1 y+\beta_2 x+\rho_0,\\
\xi^x &= \tfrac{1}{2}\alpha_1(t^2+x^2-y^2)+(\delta+\alpha_0 t+\alpha_2 y)x-\beta_0 y+\beta_2 t+\rho_1,\\
\xi^y &= \tfrac{1}{2}\alpha_2(t^2-x^2+y^2)+(\delta+\alpha_0 t+\alpha_1 x)y+\beta_0 x+\beta_1 t+\rho_2,\\
\eta^u &= \lambda u+\varphi-\tfrac{1}{2}(\beta_2+\alpha_0 x+\alpha_1 t)u\\
&\quad+\tfrac{1}{2}(\alpha_2(x-t)-(\alpha_0+\alpha_1)y-(\beta_0+\beta_1))v,\\
\eta^v &= \lambda v+\psi+\tfrac{1}{2}(\beta_2+\alpha_0 x+\alpha_1 t)v\\
&\quad-\tfrac{1}{2}(\alpha_2(x+t)+(\alpha_0-\alpha_1)y+(\beta_0-\beta_1))u,
\end{aligned}$$

where α_i, β_i, ρ_i, $i=0,1,2$, and δ are arbitrary constants whereas λ, φ, and ψ are arbitrary smooth functions of the independent variables t, x, and y. Therefore, the general form of the infinitesimal operator is

$$X=\chi+\alpha_i K^i+\beta_i J^i+\delta D+\rho_i P^i, \tag{4}$$

where α_i, β_i, ρ_i, $i=0,1,2$, and δ are arbitrary constants and

$$\begin{aligned}
&\chi=(\lambda u+\varphi)\partial_u+(\lambda v+\psi)\partial_v,\\
&P^0=\partial_t,\quad P^1=\partial_x,\quad P^2=\partial_y,\\
&D=t\partial_t+x\partial_x+y\partial_y,\\
&J^0=x\partial_y-y\partial_x-\frac{1}{2}v\partial_u+\frac{1}{2}u\partial_v,\\
&J^1=t\partial_y+y\partial_t-\frac{1}{2}v\partial_u-\frac{1}{2}u\partial_v,\\
&J^2=t\partial_x+x\partial_t-\frac{1}{2}u\partial_u+\frac{1}{2}v\partial_v,\\
&K^0=\frac{1}{2}(t^2+x^2+y^2)\partial_t+tx\partial_x+ty\partial_y-\frac{1}{2}(xu+yv)\partial_u+\frac{1}{2}(xv-yu)\partial_v,\\
&K^1=tx\partial_t+\frac{1}{2}(t^2+x^2-y^2)\partial_x+xy\partial_y-\frac{1}{2}(tu+yv)\partial_u+\frac{1}{2}(yu+tv)\partial_v,\\
&K^2=ty\partial_t+xy\partial_x+\frac{1}{2}(t^2-x^2+y^2)\partial_y+\frac{1}{2}(x-t)v\partial_u-\frac{1}{2}(x+t)u\partial_v.
\end{aligned}$$

[1]The computations were verified using GeM software package for computation of symmetries and conservation laws of differential equations [10].

(5)

The usual summation convention, i.e., summation over repeated indices, is used in (4). The nonzero commutation relations are

$$[P^i, K^i] = D,\ [P^0, J^1] = P^2,\ [P^0, J^2] = P^1,\ [P^1, J^0] = P^2,$$
$$[P^1, J^2] = P^0,\ [P^2, J^0] = -P^1,\ [P^2, J^1] = P^0,\ [J^0, J^1] = J^2,$$
$$[J^0, J^2] = -J^0,\ [J^1, J^2] = -J^1,\ [P^0, K^1] = J^2,\ [P^0, K^2] = J^1,$$
$$[P^1, K^0] = J^2,\ [P^1, K^2] = -J^0,\ [P^2, K^0] = J^1,\ [P^2, K^1] = J^0,$$

where $i = 0, 1, 2$.

Then the remaining determining equations, which involve the functions F, G and their first-order partial derivatives with respect to u and v, take the form

$$\begin{aligned}
&\left[\left(\lambda-\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)u-\tfrac{1}{2}\left(\alpha_2(t-x)+(\alpha_0+\alpha_1)y+\beta_0+\beta_1\right)v+\varphi\right]F_u\\
&\quad+\left[\tfrac{1}{2}\left(-\alpha_2(x+t)+(\alpha_1-\alpha_0)y+\beta_0-\beta_1\right)u+\left(\lambda+\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)v+\psi\right]F_v\\
&\quad+\tfrac{1}{2}\left[\alpha_2(t-x)+(\alpha_1+\alpha_0)y+\beta_0+\beta_1\right]G\\
&\quad+\left[(\alpha_0+\tfrac{1}{2}\alpha_1)t+(\alpha_1+\tfrac{1}{2}\alpha_0)x+\alpha_2y+\tfrac{1}{2}\beta_2+\delta-\lambda\right]F\\
&\quad+(\lambda_y+\alpha_2)u-(\lambda_t+\lambda_x+\alpha_1+\alpha_0)v-\psi_t-\psi_x+\varphi_y=0,\\
&\left[\left(\lambda-\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)u-\tfrac{1}{2}\left(\alpha_2(t-x)+(\alpha_0+\alpha_1)y+\beta_0+\beta_1\right)v+\varphi\right]G_u\\
&\quad+\left[\tfrac{1}{2}\left(-\alpha_2(x+t)+(\alpha_1-\alpha_0)y+\beta_0-\beta_1\right)u+\left(\lambda+\tfrac{1}{2}(\beta_2+\alpha_0x+\alpha_1t)\right)v+\psi\right]G_v\\
&\quad+\tfrac{1}{2}\left[\alpha_2(x+t)-(\alpha_1-\alpha_0)y-\beta_0+\beta_1\right]F\\
&\quad+\left[(\alpha_0-\tfrac{1}{2}\alpha_1)t+(\alpha_1-\tfrac{1}{2}\alpha_0)x+\alpha_2y-\tfrac{1}{2}\beta_2+\delta-\lambda\right]G\\
&\quad+(\lambda_t-\lambda_x-\alpha_1+\alpha_0)u-(\lambda_y+\alpha_2)v+\varphi_t-\varphi_x-\psi_y=0.
\end{aligned} \tag{6}$$

These equations are called classifying equations. The main difficulty of group classification problem is that they should be solved for remaining uncertainties in the coefficients of X and the arbitrary elements of the class simultaneously.

2.2. The kernel algebra

In order to find Lie symmetries which are admitted by any PDE system (2) we should split classifying equations (6) with respect to the functions F, G and their derivatives. This results in the conditions $\alpha_i = \beta_j = 0$, $i, j = 0, 1, 2$, and $\delta = \lambda = \varphi = \psi = 0$, ρ^k, $k = 0, 1, 2$, are arbitrary constants. Therefore, the following statement is true.

Lemma 1 *The kernel of the maximal Lie invariance algebras of systems of equations from class* (2) *coincides with the three-dimensional algebra*

$$A^{\ker} = \langle P^0,\ P^1,\ P^2\rangle.$$

It means that any PDE system from class (2) is invariant with respect to translations by t, x and y.

2.3. One-dimensional extension of $A^{\max}$

As classifying equations are quite complicated we at first consider extension of the kernel algebra $A^{\ker}$ on one symmetry generator. For this purpose we take the general form of the admitted Lie symmetry operator (4) and require that X and operators from $A^{\ker}$ form a Lie algebra, i.e., we require

$$[A^{\ker}, X] \in \langle A^{\ker}, X\rangle.$$

This condition implies the equalities

$$\begin{aligned}
[P^0, X] &= \alpha_0 D + \alpha_1 J^2 + \alpha_2 J^1 + \chi_t = aX,\\
[P^1, X] &= \alpha_0 J^2 + \alpha_1 D - \alpha_2 J^0 + \chi_x = bX,\\
[P^2, X] &= \alpha_0 J^1 + \alpha_1 J^0 + \alpha_2 D + \chi_y = cX,
\end{aligned}$$

which result in the two possibilities for X

1. $(a, b, c) = (0, 0, 0) \quad \Rightarrow \quad X = \delta D + \beta_i J^i + \chi,$
2. $(a, b, c) \neq (0, 0, 0) \quad \Rightarrow \quad X = e^{at+bx+cy}\chi.$

Here a, b and c are arbitrary real constants, $\chi = (\lambda u + \varphi)\partial_u + (\lambda v + \psi)\partial_v$ with λ, φ, and ψ being arbitrary constants.

Case I. If $X = \delta D + \beta_i J^i + (\lambda u + \varphi)\partial_u + (\lambda v + \psi)\partial_v$, where λ, φ, and ψ are arbitrary constants, then equations (6) take the form

$$\begin{aligned}
&[(2\lambda - \beta_2)u - (\beta_0 + \beta_1)v + 2\varphi]\, F_u + [(\beta_0 - \beta_1)u\\
&\qquad + (2\lambda + \beta_2)v + 2\psi]\, F_v + (\beta_1 + \beta_0)\, G + (2\delta - 2\lambda + \beta_2)\, F = 0,\\
&[(2\lambda - \beta_2)u - (\beta_0 + \beta_1)v + 2\varphi]\, G_u + [(\beta_0 - \beta_1)u\\
&\qquad + (2\lambda + \beta_2)v + 2\psi]\, G_v + (2\delta - 2\lambda - \beta_2)\, G + (\beta_1 - \beta_0)\, F = 0.
\end{aligned} \tag{7}$$

This PDE system admits the point equivalence transformations

$$\begin{aligned}
u &= a_1\tilde{u} + b_1\tilde{v} + c_1, \qquad & v &= a_2\tilde{u} + b_2\tilde{v} + c_2,\\
F &= a_1\tilde{F} + b_1\tilde{G}, & G &= a_2\tilde{F} + b_2\tilde{G}.
\end{aligned}$$

Here a_i, b_i and c_i, $i = 1, 2$, are arbitrary constants with $\Delta = a_1b_2 - b_1a_2 \neq 0$. The constant parameters appearing in system (7) are changed under the action of these transformations as follows: $\tilde{\lambda}d = \tilde{d}\lambda$,

$$\tilde{\beta}_0 = \frac{1}{2\Delta}\frac{\tilde{d}}{d}\left((\beta_0 - \beta_1)(a_1^2 + b_1^2) + (\beta_0 + \beta_1)(a_2^2 + b_2^2) + 2\beta_2(a_1a_2 + b_1b_2)\right),$$

$$\tilde{\beta}_1 = -\frac{1}{2\Delta}\frac{\tilde{d}}{d}\left((\beta_0-\beta_1)(a_1^2-b_1^2)+(\beta_0+\beta_1)(a_2^2-b_2^2)+2\beta_2(a_1a_2-b_1b_2)\right),$$

$$\tilde{\beta}_2 = \frac{1}{\Delta}\frac{\tilde{d}}{d}\left((\beta_0-\beta_1)a_1b_1+(\beta_0+\beta_1)a_2b_2+\beta_2(a_1b_2+b_1a_2)\right),$$

$$\tilde{\varphi} = -\frac{1}{2\Delta}\frac{\tilde{d}}{d}\,[(\beta_0-\beta_1)b_1c_1+(\beta_0+\beta_1)b_2c_2+\beta_2(b_1c_2+c_1b_2)$$
$$+2\,\lambda(b_1c_2-c_1b_2)+2b_1\psi-2b_2\varphi]\,,$$

$$\tilde{\psi} = \frac{1}{2\Delta}\frac{\tilde{d}}{d}\,[(\beta_0-\beta_1)a_1c_1+(\beta_0+\beta_1)a_2c_2+\beta_2(a_1c_2+c_1a_2)$$
$$+2\,\lambda(a_1c_2-c_1a_2)+2a_1\psi-2a_2\varphi]\,,$$

where $\Delta = a_1b_2 - b_1a_2 \neq 0$. Considering the possibilities of simplifying the coefficients we obtain that the nonzero triple $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)$ has only three inequivalent values depending on the sign of $D = b_2^2 + b_1^2 - b_0^2$:

$$(0,0,1)\ \text{ if }\ D>0, \qquad (1,1,0)\ \text{ if }\ D=0, \qquad (1,0,0)\ \text{ if }\ D<0.$$

Therefore, it is sufficient to consider four inequivalent forms of the constant tuple $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\delta}, \tilde{\lambda}, \tilde{\varphi}, \tilde{\psi})$, namely,

$$\begin{array}{ll} 1.\quad (0,0,1,\delta',\lambda',0,0), & 2.\quad (1,0,0,\delta',\lambda',0,0),\\ 3.\quad (1,1,0,\delta',\lambda',0,0), & 4.\quad (1,1,0,\delta',0,\varphi',\psi') \end{array}$$

and the fifth possibility arises when $\beta_0 = \beta_1 = \beta_2 = 0$, namely, the tuple

$$5.\quad (0,0,0,\delta',\lambda',0,0)$$

should also be considered.

Consider the first case, where $\beta_0 = \beta_1 = \varphi = \psi = 0$, $\beta_2 = 1$, δ and λ are arbitrary constants. Then system (7) becomes

$$\begin{aligned}
&(2\lambda-1)uF_u+(2\lambda+1)vF_v+(2\delta-2\lambda+1)\,F=0,\\
&(2\lambda-1)uG_u+(2\lambda+1)vG_v+(2\delta-2\lambda-1)\,G=0.
\end{aligned} \tag{8}$$

If $\lambda \neq 1/2$, then the general solution of (8) is $F = u^{1+\frac{2\delta}{1-2\lambda}}\,\mathcal{F}(vu^{\frac{1+2\lambda}{1-2\lambda}})$, $G = u^{1+\frac{2\delta-2}{1-2\lambda}}\,\mathcal{G}(vu^{\frac{1+2\lambda}{1-2\lambda}})$ (Case 1 of Table 1). Here and below the functions $\mathcal{F}$ and $\mathcal{G}$ are arbitrary smooth functions of their variables.

If $\lambda = \frac{1}{2}$, then $F = v^{-\delta}\,\mathcal{F}(u)$, $G = v^{1-\delta}\,\mathcal{G}(u)$ (Case 2 of Table 1).

In the second case system (7) takes such a form which has no solutions over the real field $\mathbb{R}$ and can be solved over the field $\mathbb{C}$ only.

In the third and the fourth cases system (7) becomes

$$\begin{aligned}&(\lambda u - v + \varphi)\, F_u + (\lambda v + \psi)\, F_v + (\delta - \lambda)\, F + G = 0,\\&(\lambda u - v + \varphi)\, G_u + (\lambda v + \psi)\, G_v + (\delta - \lambda)\, G = 0,\end{aligned} \tag{9}$$

where either $\varphi = \psi = 0$ or $\lambda = 0$. The results of its integration are presented by Cases 3–5 of Table 1.

In the fifth case system (7) is of the form

$$uF_u + vF_v + (\delta/\lambda - 1)F = 0, \quad uG_u + vG_v + (\delta/\lambda - 1)G = 0,$$

whose general solution is $F = u^{1-\delta/\lambda}\, \mathcal{F}(v/u)$, $G = u^{1-\delta/\lambda}\, \mathcal{F}(v/u)$ (Case 6 of Table 1).

Case II. If $X = e^{at+bx+cy}\, ((\lambda u + \varphi)\partial_u + (\lambda v + \psi)\partial_v)$, where a, b, c, λ, φ, and ψ are arbitrary constants with $(a, b, c) \neq (0, 0, 0)$, then system (6) of classifying equations becomes the uncoupled system

$$\begin{aligned}&(\lambda u + \varphi)F_u + (\lambda v + \psi)F_v - \lambda F + c(\lambda u + \varphi) - (a + b)(\lambda v + \psi) = 0,\\&(\lambda u + \varphi)G_u + (\lambda v + \psi)G_v - \lambda G + (a - b)(\lambda u + \varphi) - c(\lambda v + \psi) = 0.\end{aligned} \tag{10}$$

If $\lambda \neq 0$, then the equivalence transformation $\tilde{u} = \lambda u + \varphi$, $\tilde{v} = \lambda v + \psi$, $\tilde{F} = \lambda F$, $\tilde{G} = \lambda G$ maps system (10) to the one with $\tilde{\lambda} = 1$, $\tilde{\varphi} = \tilde{\psi} = 0$, whose general solution is

$$\begin{aligned}F &= ((b + a)v - cu) \ln u + u\, \mathcal{F}(v/u)\, ,\\G &= ((b - a)u + cv) \ln u + u\, \mathcal{G}(v/u)\end{aligned}$$

(see Case 7 of Table 1).

If $\lambda = 0$, then (φ, ψ) has the following inequivalent values $(\varphi, 1)$ with $\varphi \neq 0$ and $(1, 0)$. The corresponding general solutions are

$$\begin{aligned}F &= ((b + a)/\varphi - c)\, u + \mathcal{F}(u - \varphi v)\, ,\\G &= (b - a + c/\varphi)\, u + \mathcal{G}(u - \varphi v)\, ,\end{aligned} \qquad \varphi \neq 0,\ \psi = 1,$$

$$\begin{aligned}F &= (a{+}b)v + \mathcal{F}(u),\\G &= cv + \mathcal{G}(u),\end{aligned} \qquad \varphi = 0,\ \psi = 1,$$

$$\begin{aligned}F &= -cu + \mathcal{F}(v),\\G &= (b{-}a)u + \mathcal{G}(v),\end{aligned} \qquad \varphi = 1,\ \psi = 0,$$

(Cases 8–10 of Table 1).

Let us note that the four-dimensional Lie symmetry algebras

$$\langle P^0,\, P^1,\, P^2,\, e^{at+bx+cy}(\varphi\partial_u + \psi\partial_v)\rangle, \quad (\varphi, \psi) \in \{(\varphi, 1), (1, 0)\}$$

Table 1. Lie symmetries of (2+1)-dimensional Dirac equations (2).

	Nonlinearities	Basis operators of $A^{\max}$
1 $\lambda\neq\frac12$	$F=u^{1+\frac{2\delta}{1-2\lambda}}\,\mathcal{F}(vu^{\frac{1+2\lambda}{1-2\lambda}})$, $G=u^{1+\frac{2\delta-2}{1-2\lambda}}\,\mathcal{G}(vu^{\frac{1+2\lambda}{1-2\lambda}})$	P^0, P^1, P^2, $\delta D+J^2+\lambda(u\partial_u+v\partial_v)$
2	$F=v^{-\delta}\,\mathcal{F}(u)$, $G=v^{1-\delta}\,\mathcal{G}(u)$	P^0, P^1, P^2, $\delta D+J^2+\frac12(u\partial_u+v\partial_v)$
3 $\lambda\neq0$	$F=-\frac{1}{\lambda}\,\mathrm{W}(z)G$ $+\,e^{\frac{\lambda-\delta}{\lambda}\,\mathrm{W}(z)}\mathcal{F}(\ln v+\lambda u/v)\,,$ $G=e^{\frac{\lambda-\delta}{\lambda}\,\mathrm{W}(z)}\mathcal{G}(\ln v+\lambda u/v)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\lambda(\partial_u+v\partial_v)$
4 $\psi\neq0$	$F=\dfrac{\varphi-v}{\psi}G$ $+e^{\frac{\delta}{\psi}(\varphi-v)}\,\mathcal{F}(\varphi v-\psi u-\frac12 v^2)$, $G=e^{\frac{\delta}{\psi}(\varphi-v)}\,\mathcal{G}(\varphi v-\psi u-\frac12 v^2)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\varphi\partial_u+\psi\partial_v$
5	$F=\dfrac{u}{v-\varphi}G+e^{\frac{\delta u}{v-\varphi}}\,\mathcal{F}\,(v)\,,$ $G=e^{\frac{\delta u}{v-\varphi}}\,\mathcal{G}\,(v)$	P^0, P^1, P^2, $\delta D+J^0+J^1+\varphi\partial_u$
6 $\lambda\neq0$	$F=u^{1-\delta/\lambda}\,\mathcal{F}(v/u)$, $G=u^{1-\delta/\lambda}\,\mathcal{G}(v/u)$	P^0, P^1, P^2, $\delta D+\lambda(u\partial_u+v\partial_v)$
7	$F=((b{+}a)v-cu)\ln u+u\,\mathcal{F}(v/u)\,,$ $G=((b{-}a)u+cv)\ln u+u\,\mathcal{G}(v/u)$	P^0, P^1, P^2, $e^{at+bx+cy}(u\partial_u+v\partial_v)$
8 $\varphi\neq0$	$F=((b+a)/\varphi-c)\,u+\mathcal{F}(u-\varphi v)\,,$ $G=(b-a+c/\varphi)\,u+\mathcal{G}(u-\varphi v)$	P^0, P^1, P^2, $\sigma(\omega)e^{at+bx+cy}(\varphi\partial_u+\partial_v)$
9	$F=(a{+}b)v+\mathcal{F}(u)$, $G=cv+\mathcal{G}(u)$	P^0, P^1, P^2, $\sigma(x{-}t)e^{at+bx+cy}\partial_v$
10	$F=-cu+\mathcal{F}(v)$, $G=(b{-}a)u+\mathcal{G}(v)$	P^0, P^1, P^2, $\sigma(x{+}t)e^{at+bx+cy}\partial_u$

Here δ, λ, φ, ψ, a, b and c are constants with $a^2+b^2+c^2\neq0$, $\omega=\varphi^2(t{+}x){+}2\varphi y{+}t{-}x$; σ is arbitrary smooth nonvanishing function of the indicated variable. $\mathrm{W}(z)=\mathrm{LambertW}(z)$ [11], where $z=-\lambda\frac{u}{v}e^{-\lambda\frac{u}{v}}$.

are not maximal for nonlinearities presented in Cases 8–10. The corresponding systems (2) admit infinite-dimensional Lie symmetry algebras with basis operators adduced in Table 1.

Therefore we have found all nonlinearities F and G, for which (2+1)-dimensional real Dirac equations admit extension on one-symmetry generator. It should be noted that in most cases the maximal Lie invariance algebra becomes four-dimensional (Cases 1–7), but sometimes the extension operator appears to involve an arbitrary function and in this case the maximal Lie invariance algebra becomes infinite-dimensional (Cases 8–10).

Note 1. Table 1 consists some cases that are equivalent with respect to point transformations. Thus, Cases 9 and 10 are mapped to each other by the transformation

$$t \mapsto -t, \quad x \mapsto x, \quad y \mapsto -y, \quad u \mapsto v, \quad v \mapsto u, \quad F \mapsto G, \quad G \mapsto F,$$

that belongs to the equivalence group of class (2). The same transformation maps Case 2 to Case 1 with $\lambda = -1/2$.

Note 2. The Lie invariance algebras adduced in Table 1 are maximal Lie invariance algebras for arbitrary values of constants and functions appearing in corresponding nonlinearities F and G. For certain values of the parameters these algebras will not be maximal. For example, let $\varphi = 0$, $\mathcal{F}_v = \mathcal{G}_v = 0$ in Case 5 of Table 1, then the system (2) takes the form

$$\begin{aligned} v_t + v_x - u_y &= \left(\frac{u}{v}\kappa_1 + \kappa_2\right) e^{\frac{\delta u}{v}}, \\ u_t - u_x - v_y &= -\kappa_1 e^{\frac{\delta u}{v}}, \end{aligned}$$

where κ_1 and κ_2 are constants, $\kappa_1^2 + \kappa_2^2 \neq 0$. It admits additional Lie symmetry generated by the operator $D + u\partial_u + v\partial_v$. The maximal Lie invariance algebra is five-dimensional in this case.

3. Reduction procedure

In this section we present a couple of examples of finding exact solutions of NLDEs from class (2) via reduction method. This technique is well known and quite algorithmic (see, e.g., [6, 7]). Consider Case 6 of Table 1 with $\delta = \lambda \neq 0$, namely, PDE systems (2) of the form

$$\begin{aligned} v_t + v_x - u_y &= \mathcal{F}(v/u), \\ u_t - u_x - v_y &= -\mathcal{G}(v/u), \end{aligned} \tag{11}$$

admitting the Lie invariance algebra

$$\mathfrak{g} = \langle P^0, \ \ P^1, \ \ P^2, \ \ D + u\partial_u + v\partial_v \rangle.$$

Using one-dimensional subalgebras of $\mathfrak{g}$ we can reduce family of PDE systems (11) to PDE system in (1+1) dimensions. Consider, for example, the one-dimensional subalgebra $\langle D+u\partial_u+v\partial_v\rangle$ of $\mathfrak{g}$. This subalgebra belongs to the optimal system of subalgebras of $\mathfrak{g}$ (a procedure of finding the optimal system is well described in [7] and classification of subalgebras of real three- and four-dimensional algebras can be found in [12]). The substitutions $u=t\,U(z,w)$, $v=t\,V(z,w)$, where $z=x/t$, $w=y/t$, reduce (11) to the PDE system

$$\begin{aligned}(1-z)V_z-wV_w-U_w+V&=\mathcal{F}(V/U),\\ (1+z)U_z+wU_w+V_w-U&=\mathcal{G}(V/U)\end{aligned}\tag{12}$$

in (1+1) dimensions.

We have found simple particular solutions of this system for arbitrary $\mathcal{F}=\pm\mathcal{G}$. It is

$$U=\mp V=(w\mp 1)C_1\mp\mathcal{F}(\mp 1),\quad \mathcal{F}=\pm\mathcal{G},$$

where C_1 is an arbitrary constant. Thus system (11) has the solutions

$$\begin{aligned}u=-v=(y-t)C_1-t\mathcal{F}(-1)\quad &\text{if}\quad \mathcal{F}=\mathcal{G},\\ u=v=(y+t)C_1+t\mathcal{F}(1)\quad &\text{if}\quad \mathcal{F}=-\mathcal{G},\end{aligned}$$

that are valid for any function $\mathcal{F}$ that is well-defined when its argument is equal to -1 (resp. to 1 in the second case).

To reduce system (11) to an ODE system a two-dimensional subalgebra should be used. Consider the two-dimensional subalgebra

$$\langle P^1+\alpha_2P^2+\alpha_0P^0,\ D+u\partial_u+v\partial_v\rangle,\quad \alpha_2,\alpha_0\in\mathbb{R}.$$

The substitutions reducing system (11) to ODE system are

$$\begin{aligned}u&=(t-\alpha_0x)U(z),\\ v&=(t-\alpha_0x)V(z),\end{aligned}\qquad\text{where}\quad z=\frac{y-\alpha_2x}{t-\alpha_0x}.$$

The corresponding ODE system is

$$\begin{aligned}((\alpha_0-1)z-\alpha_2)\,V_z-U_z+(1-\alpha_0)V&=\mathcal{F}(V/U),\\ ((\alpha_0+1)z-\alpha_2)\,U_z+V_z-(1+\alpha_0)U&=\mathcal{G}(V/U).\end{aligned}\tag{13}$$

If $\mathcal{F}=\mathcal{G}$ then the system has particular solution

$$U=-V=(\alpha_2-1+(1-\alpha_0)z)C_1-\mathcal{F}(-1)/(1-\alpha_0),\quad \alpha_0\neq 1,$$

where C_1 is an arbitrary constant. Therefore the solution of system (11) with $\mathcal{F}=\mathcal{G}$ is

$$u=-v=[(t-\alpha_0x)(\alpha_2-1)+(y-\alpha_2x)(1-\alpha_0)]\,C_1-\frac{(t-\alpha_0x)\mathcal{F}(-1)}{(1-\alpha_0)}.$$

It is valid for any $\mathcal{F}$ that is well-defined when its argument is equal to -1.

Using the reduction method exact solutions can be constructed to other NLDEs (2) with nonlinearities presented in Table 1.

4. Conclusion

The preliminary group classification of nonlinear Dirac equations in two spatial dimensions for real wave functions is carried out. Namely, all forms of nonlinearities F and G such that the corresponding NLDE (2) admits one-dimensional extension of its Lie inavariance algebra are described. The found symmetries are useful for construction of exact solutions for wide subclasses of such equations with nonlinearities presented in Table 1. A complete group classification of NLDEs (2) will be a subject of a forthcoming paper.

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