A double-well SUSY matrix model for 2D type IIA superstrings in RR background*

Fumihiko Sugino†
Okayama Institute for Quantum Physics,
Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, JAPAN

Abstract

We discuss an interpretation of a simple supersymmetric matrix model with a double-well potential as two-dimensional type IIA superstrings on a nontrivial Ramond-Ramond background. We find direct correspondence between single-trace operators in the matrix model and integrated vertex operators in type IIA theory by computing various correlation functions in both sides.

1. Introduction

Solvable matrix models for two-dimensional quantum gravity or noncritical string theory were vigorously investigated around 1990, where a main motivation was to understand nonperturbative effects in string theory [3]. While this approach has been succeeded for bosonic string theory, little has been known for superstring theory, in particular which possesses target-space supersymmetry (SUSY). We would like to consider (solvable) matrix models describing superstring theory with target-space SUSY. We hope our analysis helpful to understand nonperturbative dynamics of matrix models of super Yang-Mills type for critical superstring theory [4].

2. Double-well SUSY matrix model

Ref. [5] discussed a following simple matrix model:

$$S = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2-\mu^2) + \bar{\psi}(\phi\psi+\psi\phi)\right], \tag{1}$$

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† e-mail address: fumihiko_sugino@pref.okayama.lg.jp

where B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ Grassmann-odd matrices. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

from which one can see that they are nilpotent: $Q^2 = \bar{Q}^2 = 0$. After integrating out B, we have a scalar potential of a double-well shape: $\frac{1}{2}(\phi^2 - \mu^2)^2$. A large-$N$ saddle point equation for the eigenvalue distribution of the matrix ϕ: $\rho(x) \equiv \frac{1}{N}\mathrm{tr}\,\delta(x-\phi)$ reads

$$\int dy\,\rho(y)\,\mathrm{P}\frac{1}{x-y} + \int dy\,\rho(y)\,\mathrm{P}\frac{1}{x+y} = x^3 - \mu^2 x. \tag{4}$$

Its solution with filling fraction (ν_+, ν_-) is given by

$$\rho(x) = \begin{cases} \frac{\nu_+}{\pi}\, x\,\sqrt{(x^2-a^2)(b^2-x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi}\, |x|\,\sqrt{(x^2-a^2)(b^2-x^2)} & (-b < x < -a) \end{cases} \tag{5}$$

with $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$. The filling fractions satisfying $\nu_+ + \nu_- = 1$ indicate that $\nu_+ N$ ($\nu_- N$) eigenvalues are around the right (left) minimum of the double-well. The solution exists for $\mu^2 > 2$. The large-N free energy and $\left\langle \frac{1}{N}\mathrm{tr}\,B^n \right\rangle$ $(n = 1, 2, \cdots)$ evaluated at the solution turn out to all vanish. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parametrized by (ν_+, ν_-) at large N. Note that the edges of the support a and b are independent of $\nu_\pm$. It is considered to be a characteristic feature of SUSY matrix models, not observed in bosonic double-well matrix models [6].

There exists a solution having support of a single interval $x \in [-c, c]$ for $\mu^2 < 2$ [7]:

$$\rho(x) = \frac{1}{2\pi}\left(x^2 - \mu^2 + \frac{c^2}{2}\right)\sqrt{c^2 - x^2} \tag{6}$$

with $c = \sqrt{\frac{2}{3}}\left(\mu^2 + \sqrt{\mu^4 + 12}\right)^{1/2}$. Positivity of $\rho(x)$ yields the condition $\mu^2 < 2$. This solution gives nonzero values of $\left\langle \frac{1}{N}\mathrm{tr}B \right\rangle$ and of the large-N free energy, showing that SUSY is broken. We observed that the third derivative of the free energy with respect to μ^2 is not continuous at $\mu^2 = 2$. The transition between the SUSY phase ($\mu^2 > 2$) and the SUSY broken phase ($\mu^2 < 2$) is of the third order.

In what follows we will compute various correlation functions at the saddle point (5) and find new logarithmic critical behavior as $\mu^2 \to 2 + 0$. Based on the result, we will discuss correspondence between the matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond (RR) background. The logarithmic critical behavior is somewhat reminiscent of the $c = 1$ matrix model which is a matrix quantum mechanics of a single matrix variable [8]. The Penner model is known as a zero-dimensional matrix model exhibiting the same critical behavior as the matrix quantum mechanics [9]. [1] The partition function is given by

$$Z_{\text{Penner}} = \int d^{N^2}M \, \exp\left[Nt \, \text{tr}\{M + \ln(1 - M)\}\right], \tag{7}$$

where the double scaling limit is taken as $n \to \infty$, $t \to -1$ with $N(1 + t)$ fixed. It describes noncritical string theory propagating on a two-dimensional target space: (Liouville direction) $\times$ (S^1 with self-dual radius). So, it is expected that our matrix model can be regarded as a SUSY version of the Penner model and describes two-dimensional superstring theory with SUSY on the target space (Liouville direction) $\times$ (S^1 with self-dual radius). Indeed, two-dimensional type II superstring theory with the identical target space is constructed [11, 12, 13, 14], where target space SUSY exists only at the self-dual radius of the circle.

Our matrix model is interpreted as the $O(n)$ model on a random surface with $n = -2$, whose critical behavior is described by the $c = -2$ topological gravity [15]. The partition function after B, ψ and $\bar{\psi}$ integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H = \phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. Ref. [16] discusses that the difference of the integration region has only effects which are nonperturbative in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion.

The Nicolai mapping changes the operators $\frac{1}{N}\text{tr}\,\phi^{2n}$ $(n = 1, 2, \cdots)$ to regular operators $\frac{1}{N}\text{tr}\,H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix (the $c = -2$ topological gravity) at least perturbatively in $1/N$. However, the operators $\frac{1}{N}\text{tr}\,\phi^{2n+1}$ $(n = 0, 1, 2, \cdots)$ are mapped to $\pm\frac{1}{N}\text{tr}\,H^{n+1/2}$ that are singular at the origin. They are not observables in the $c = -2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\text{tr}\,\phi^{2n}$ in the original setting (1). In the next section, we will see that correlation functions among operators

$$\frac{1}{N}\text{tr}\,\phi^{2n+1}, \qquad \frac{1}{N}\text{tr}\,\psi^{2n+1}, \qquad \frac{1}{N}\text{tr}\,\bar{\psi}^{2n+1} \qquad (n = 0, 1, 2, \cdots) \tag{8}$$

exhibit logarithmic singular behavior of powers of $\ln(\mu^2 - 2)$ at the planar topology.

[1] Also is the normal matrix model [10], which corresponds to $c = 1$ noncritical strings on S^1 with a general radius.

In considering correspondence of the matrix model to superstring theory, the following observation will be helpful. Suppose ψ and $\bar{\psi}$ are regarded as target-space fermions in the corresponding superstring theory. Namely, ψ is interpreted as an operator in the (NS, R) sector and $\bar{\psi}$ in the (R, NS) sector in the RNS formalism. Then, under the so-called $(-1)^{\mathbf{F}_L}$ and $(-1)^{\mathbf{F}_R}$ transformations changing the signs of operators in the left-moving Ramond sector and those in the right-moving Ramond sector respectively, they transform as

$$(-1)^{\mathbf{F}_L}: \qquad \psi \to \psi, \qquad \bar{\psi} \to -\bar{\psi}, \tag{9}$$
$$(-1)^{\mathbf{F}_R}: \qquad \psi \to -\psi, \qquad \bar{\psi} \to \bar{\psi}. \tag{10}$$

In order for the matrix model action (1) to be invariant under the transformations, B and ϕ should transform as

$$(-1)^{\mathbf{F}_L}: \qquad B \to B, \qquad \phi \to -\phi, \tag{11}$$
$$(-1)^{\mathbf{F}_R}: \qquad B \to B, \qquad \phi \to -\phi. \tag{12}$$

This indicates that B corresponds to an operator in the (NS, NS) sector, and ϕ in the (R, R) sector.

3. Correlation functions

3.1. Planar one-point functions

The planar one-point function $\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n \right\rangle_0$ $(n = 1, 2, \cdots)$ are computed as

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n \right\rangle_0 &= \int dx\, x^n \rho(x) \\ &= (\nu_+ + (-1)^n \nu_-)(2+\mu^2)^{n/2} F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{4}{2+\mu^2}\right),\end{aligned} \tag{13}$$

where the suffix "0" in the left hand side indicates the planar contribution. For n even, the expression is reduced to a polynomial of μ^2 giving nonsingular behavior as expected from the $c = -2$ topological gravity. On the other hand, when μ^2 is odd, it exhibits logarithmic singular behavior as $\mu^2 \to 2 + 0$:

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} \right\rangle_0 \sim (\nu_+ - \nu_-)\frac{2^{k+2}}{\pi}\frac{(2k+1)!!}{(k+2)!}\,\omega^{k+2}\ln\omega \tag{14}$$

with $\omega \equiv \frac{1}{4}(\mu^2 - 2)$. The symbol "$\sim$" denotes equality up to additive less singular terms. Explicit form for a first few expectation values reads

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi \right\rangle_0 = (\nu_+ - \nu_-)\left[\frac{64}{15\pi} + \frac{16}{3\pi}\omega + \frac{2}{\pi}\omega^2\ln\omega + \mathcal{O}(\omega^2)\right],$$

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,\phi^3\right\rangle_0 &= (\nu_+ - \nu_-)\left[\frac{1024}{105\pi} + \frac{128}{5\pi}\omega + \frac{16}{\pi}\omega^2 + \frac{4}{\pi}\omega^3\ln\omega + \mathcal{O}(\omega^3)\right],\\ \left\langle \frac{1}{N}\mathrm{tr}\,\phi^5\right\rangle_0 &= (\nu_+ - \nu_-)\left[\frac{8192}{315\pi} + \frac{2048}{21\pi}\omega + \frac{128}{\pi}\omega^2 + \frac{160}{3\pi}\omega^3 + \frac{10}{\pi}\omega^4\ln\omega\right.\\ &\qquad\left. + \mathcal{O}(\omega^4)\right],\\ &\cdots\cdots\cdots\ . \end{aligned} \tag{15}$$

Matrix models can be seen as a sort of "lattice models" for string theory. In the hypergeometric function $F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{1}{1+\omega}\right)$ for n: odd, the logarithmic singular terms can be regarded as universal parts relevant to "continuum physics", whereas polynomials of ω as nonuniversal "lattice artifacts".

3.2. Eigenvalue distribution with source

In computing higher-point correlators $\left\langle \prod_{i=1}^K \frac{1}{N}\mathrm{tr}\,\phi^{n_i}\right\rangle_{C,0}$ at the vacuum with general filling fractions (ν_+, ν_-), it is useful to reduce them to those at the vacuum with $(\nu_+, \nu_-) = (1, 0)$. We can show

$$\left\langle \prod_{i=1}^K \frac{1}{N}\mathrm{tr}\,\phi^{n_i}\right\rangle_{C,0}^{(\nu_+,\nu_-)} = (\nu_+ - \nu_-)^\sharp \left\langle \prod_{i=1}^K \frac{1}{N}\mathrm{tr}\,\phi^{n_i}\right\rangle_{C,0}^{(1,0)} \tag{16}$$

up to $K = 3$, by explicit calculations. Here, the suffix "C" means taking the connected correlator. The superscripts (ν_+, ν_-) and $(1, 0)$ are put to clarify the filling fractions of the vacua at which the expectation values are evaluated, and $\sharp$ counts the number of odd integers in $\{n_1, \cdots, n_K\}$.

In order to obtain higher-point correlators of $\frac{1}{N}\mathrm{tr}\,\phi^p$ $(p = 1, 2, \cdots)$, we introduce source terms $\sum_{p=1}^\infty j_p \mathrm{tr}\,\phi^p$ to the partition function:

$$Z_{j_k} = \int d^{N^2}\phi\, e^{-N\mathrm{tr}\left[\frac{1}{2}(\phi^2-\mu^2)^2 - \sum_{p=1}^\infty j_p\phi^p\right]} \det\left(\phi\otimes\mathbf{1} + \mathbf{1}\otimes\phi\right). \tag{17}$$

In the large-N limit, the eigenvalue distribution $\rho_j(x)$ satisfies the saddle point equation

$$\int dy\, \rho_j(y)\left(\mathrm{P}\frac{1}{x-y} + \mathrm{P}\frac{1}{x+y}\right) = x^3 - \mu^2 x - \sum_{p=1}^\infty \frac{p j_p}{2}\, x^{p-1}. \tag{18}$$

Let us consider the case of the filling fractions $(1, 0)$ with the support of $\rho_j(x)$ $[a_j, b_j]$ $(0 < a_j < b_j)$. We change variables as

$$x^2 = A + B\xi, \quad y^2 = A + B\eta \qquad \text{with} \qquad A \equiv \frac{a_j^2 + b_j^2}{2}, \quad B \equiv \frac{b_j^2 - a_j^2}{2}, \tag{19}$$

and put $\tilde{\rho}(\eta) \equiv \frac{B}{2y}\rho_j(y)$, to simplify (18) as

$$\frac{1}{B}\int_{-1}^{1} d\eta\, \tilde{\rho}(\eta)\, \mathrm{P}\frac{1}{\xi-\eta} = \frac{1}{2}(A-\mu^2+B\xi) - \sum_{p=1}^{\infty}\frac{pj_p}{4}\,(A+B\xi)^{\frac{p}{2}-1} \tag{20}$$

for $\xi \in [-1,1]$, where $\tilde{\rho}$ is normalized by $\int_{-1}^{1} d\eta\tilde{\rho}(\eta) = 1$.
We act $\int_{-1}^{1} d\xi\,\sqrt{1-\xi^2}\,\mathrm{P}\frac{1}{\zeta-\xi}$ to both sides of (20), and apply the formula

$$\int_{-1}^{1} dy\,\sqrt{1-y^2}\,\mathrm{P}\frac{1}{x-y}\,\mathrm{P}\frac{1}{u-y} = -\pi + \pi^2\sqrt{1-u^2}\,\delta(u-x) \tag{21}$$

for $x, u \in [-1,1]$. Then

$$\tilde{\rho}(\zeta) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\zeta^2}}\Bigg[2-(A-\mu^2)B\zeta - B^2\left(\zeta^2-\frac{1}{2}\right) + \sum_{p=1}^{\infty} pj_p\,\frac{B}{2\pi}\int_{-1}^{1} d\xi\sqrt{1-\xi^2}\,\mathrm{P}\frac{1}{\zeta-\xi}\,(A+B\xi)^{\frac{p}{2}-1}\Bigg] \tag{22}$$

is obtained. The condition $\tilde{\rho}(\zeta=\pm 1)=0$ determines A and B as

$$A = \mu^2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}(A+B)^{\frac{p}{2}-1}F\left(-\frac{p}{2}+1, \frac{1}{2}, 1; \frac{2B}{A+B}\right), \tag{23}$$

$$B = 2\left[1+\sum_{p=1}^{\infty}\frac{j_p}{4}\,\frac{p}{2}\left(\frac{p}{2}-1\right)B^2(A+B)^{\frac{p}{2}-2}F\left(-\frac{p}{2}+2, \frac{3}{2}, 3; \frac{2B}{A+B}\right)\right]^{1/2}, \tag{24}$$

from which A and B are obtained iteratively with respect to $\{j_p\}$. Up to the first order of $\{j_p\}$,

$$A = \mu^2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}(2+\mu^2)^{\frac{p}{2}-1}F\left(-\frac{p}{2}+1, \frac{1}{2}, 1; \frac{4}{2+\mu^2}\right) + \mathcal{O}(j^2), \tag{25}$$

$$B = 2 + \sum_{p=1}^{\infty} j_p\,\frac{p}{2}\left(\frac{p}{2}-1\right)(2+\mu^2)^{\frac{p}{2}-2}F\left(-\frac{p}{2}+2, \frac{3}{2}, 3; \frac{4}{2+\mu^2}\right) + \mathcal{O}(j^2), \tag{26}$$

where $\mathcal{O}(j^2)$ means a quantity of the quadratic order of $\{j_p\}$.

3.3. Planar two-point functions (bosons)

Let us express the planar expectation value of $\mathcal{O}$ under the partition function with the source terms (17) as $\langle \mathcal{O} \rangle_0^{(j)}$. The cylinder amplitude at the vacuum with the filling fractions $(1,0)$ is given as

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi^p \,\frac{1}{N}\mathrm{tr}\,\phi^q \right\rangle_{C,0}^{(1,0)} = \frac{\partial}{\partial j_p} \left\langle \frac{1}{N}\mathrm{tr}\,\phi^q \right\rangle_0^{(j)} \Bigg|_{\{j_p\}=0} = \frac{\partial}{\partial j_p} \int_{-1}^{1} d\zeta \,(A+B\zeta)^{\frac{q}{2}} \tilde{\rho}(\zeta) \Bigg|_{\{j_p\}=0}. \quad (27)$$

Combining (27) and (16) leads to the result for general filling fractions. In what follows, we omit the superscript (ν_+, ν_-) of the correlators when there is no possible confusion. It turns out that the amplitudes take quadratic forms of the hypergeometric functions. When p and q are even, they are polynomials of ω independent of $(\nu_+ - \nu_-)$, which is expected from the $c = -2$ topological gravity. When p and q are odd and even respectively,

$$\left\langle \Phi_{2k+1} \frac{1}{N}\mathrm{tr}\,\phi^{2\ell} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)(\mathrm{const.})\omega^{k+1} \ln \omega. \quad (28)$$

When p and q are odd,

$$\langle \Phi_{2k+1} \Phi_{2\ell+1} \rangle_{C,0} \sim -(\nu_+ - \nu_-)^2 \frac{1}{2\pi^2} \frac{1}{k+\ell+1} \frac{(2k+1)!}{(k!)^2} \frac{(2\ell+1)!}{(\ell!)^2} \times \omega^{k+\ell+1} (\ln \omega)^2. \quad (29)$$

Here, in order to subtract nonuniversal contributions in the form of the product: (nonuniversal part) $\times$ (universal part), we took a basis of the odd-power operators (operator mixing)

$$\Phi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\nu_+ - \nu_-) \sum_{i=1}^{k} \alpha_{2k+1,2i}(\omega) \frac{1}{N}\mathrm{tr}\,\phi^{2i} \quad (30)$$

with $\alpha_{2k+1,2i}(\omega)$ being a regular function at $\omega = 0$. For example, we can explicitly construct the basis for the first three operators by considering $\langle \Phi_1 \Phi_1 \rangle_{C,0}$, $\langle \Phi_1 \Phi_3 \rangle_{C,0}$, $\cdots$, $\langle \Phi_5 \Phi_5 \rangle$:

$$\begin{aligned} \Phi_1 &= \frac{1}{N}\mathrm{tr}\,\phi, \\ \Phi_3 &= \frac{1}{N}\mathrm{tr}\,\phi^3 - (\nu_+ - \nu_-)\frac{4}{\pi}\left(1 + \bar{\alpha}_{3,2}^{(1)}\omega + \mathcal{O}(\omega^2)\right) \frac{1}{N}\mathrm{tr}\,\phi^2, \\ \Phi_5 &= \frac{1}{N}\mathrm{tr}\,\phi^5 - (\nu_+ - \nu_-)\frac{4}{\pi}\left(1 + \bar{\alpha}_{5,4}^{(1)}\omega + \mathcal{O}(\omega^2)\right) \frac{1}{N}\mathrm{tr}\,\phi^4, \\ &\quad -(\nu_+ - \nu_-)\frac{8}{3\pi}\left(1 + 3(1-\bar{\alpha}_{5,4}^{(1)})\omega + \mathcal{O}(\omega^2)\right) \frac{1}{N}\mathrm{tr}\,\phi^2, \end{aligned} \quad (31)$$

where $\bar{\alpha}^{(1)}_{3,2}$ and $\bar{\alpha}^{(1)}_{5,4}$ are undertermined constants. They would be determined by considering higher operators.

Note that $(\nu_+ - \nu_-)$ corresponds to a Ramond-Ramond (RR) charge from the observation at the end of section 2. Φ_{2k+1} has a RR charge.

3.4. Planar three-point functions (bosons)

Similar procedure to the case of the two-point functions can be used in computing three-point correlation functions. It turns out that the result is expressed as cubic forms of the hypergeometric functions. The first two amplitudes become

$$\begin{aligned}\left\langle(\Phi_1)^3\right\rangle_{C,0} &= (\nu_+ - \nu_-)^3\left[\frac{1}{16\pi^3}(\ln\omega)^3 + \mathcal{O}((\ln\omega)^2)\right],\\ \left\langle(\Phi_1)^2\Phi_3\right\rangle_{C,0} &= (\nu_+ - \nu_-)^3\left[\frac{2}{\pi^3} + \frac{3}{8\pi^3}\,\omega(\ln\omega)^3 + \mathcal{O}(\omega(\ln\omega)^2)\right].\end{aligned} \tag{32}$$

3.5. Planar higher-point functions (bosons)

The results obtained for the one-, two- and three-point functions of operators Φ_{2k+1} $(k = 0, 1, 2, \cdots)$ naturally suggest the form of higher-point functions as

$$\left\langle\prod_{i=1}^{n}\Phi_{2k_i+1}\right\rangle_{C,0} \sim (\nu_+ - \nu_-)^n\,(\mathrm{const.})\,\omega^{2-\gamma+\sum_{i=1}^{n}(k_i-1)}\,(\ln\omega)^n \tag{33}$$

with $\gamma = -1$. Besides the power of logarithm $(\ln\omega)^n$, it has the standard scaling behavior with the string susceptibility $\gamma = -1$ (the same as in the $c = -2$ topological gravity) and the gravitational scaling dimension k of Φ_{2k+1}, if we identify ω with "the cosmological constant" coupled to the lowest dimensional operator on a random surface [17].

3.6. Planar two-point functions (fermions)

The simplest two-point correlator of fermions is computed as

$$\begin{aligned}\left\langle\frac{1}{N}\mathrm{tr}\,\psi\,\frac{1}{N}\mathrm{tr}\,\bar{\psi}\right\rangle_{C,0} &= \frac{1}{2}\int_\Omega dx\,\frac{1}{x}\,\rho(x)\\ &= (\nu_+ - \nu_-)\,\frac{1}{2}\,(4(1+\omega))^{-1/2}\,F\left(\frac{1}{2},\frac{3}{2},3;\frac{1}{1+\omega}\right)\\ &= (\nu_+ - \nu_-)\left[\frac{4}{3\pi} + \frac{1}{\pi}\,\omega\ln\omega + \mathcal{O}(\omega)\right] \qquad (\omega \to +0),\end{aligned} \tag{34}$$

exhibiting $\ln\omega$ singular behavior. SUSY invariance implies that this is equal to $\left\langle\frac{1}{N}\mathrm{tr}\,(iB)\,\frac{1}{N}\mathrm{tr}\,\phi\right\rangle_{C,0} = \frac{1}{4}\frac{\partial}{\partial\omega}\left\langle\frac{1}{N}\mathrm{tr}\,\phi\right\rangle_0$, interestingly which can be seen from (15).

Next, for $\left\langle \frac{1}{N}\mathrm{tr}\,\psi^3\,\frac{1}{N}\mathrm{tr}\,\bar{\psi}^3\right\rangle_{C,0}$, we should consider an operator mixing similar to the bosonic case (30). Let us take the new basis as

$$\begin{aligned} \Psi_1 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi, & \bar{\Psi}_1 &\equiv \frac{1}{N}\mathrm{tr}\,\bar{\psi}, \\ \Psi_3 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi^3 + (\text{mixing}), & \bar{\Psi}_3 &\equiv \frac{1}{N}\mathrm{tr}\,\bar{\psi}^3 + (\text{mixing}), \\ \Psi_5 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi^5 + (\text{mixing}), & \bar{\Psi}_5 &\equiv \frac{1}{N}\mathrm{tr}\,\bar{\psi}^5 + (\text{mixing}), \\ &\cdots, & &\cdots, \end{aligned} \tag{35}$$

where "mixing" means operators to be added so that

$$\left\langle \Psi_{2k+1}\bar{\Psi}_{2\ell+1}\right\rangle_{C,0} \sim \delta_{k,\ell}\,v_k\,(\nu_+ - \nu_-)^{2k+1}\omega^{2k+1}\ln\omega \tag{36}$$

with v_k constants holds for $k,\ell = 0,1$. It turns out that the choice

$$\begin{aligned} \Psi_3 &= \frac{1}{N}\mathrm{tr}\,\psi^3 + \frac{3}{\sqrt{2}}(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\mathrm{tr}\,\{(iB-\phi^2+\mu^2)\psi\}, \\ \bar{\Psi}_3 &= \frac{1}{N}\mathrm{tr}\,\bar{\psi}^3 + \frac{3}{\sqrt{2}}(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\mathrm{tr}\,\{(iB-\phi^2+\mu^2)\bar{\psi}\} \end{aligned} \tag{37}$$

or

$$\begin{aligned} \Psi_3 &= \frac{1}{N}\mathrm{tr}\,\psi^3 - \frac{3}{\sqrt{2}}(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\mathrm{tr}\,\{(iB-\phi^2+\mu^2)\psi\}, \\ \bar{\Psi}_3 &= \frac{1}{N}\mathrm{tr}\,\bar{\psi}^3 - \frac{3}{\sqrt{2}}(1+\omega+\mathcal{O}(\omega^2))\frac{1}{N}\mathrm{tr}\,\{(iB-\phi^2+\mu^2)\bar{\psi}\} \end{aligned} \tag{38}$$

does the job (36) with $v_0 = \frac{1}{\pi}$ and $v_1 = \frac{6}{\pi}$.

The result (36) tells us that Ψ_{2k+1} and $\bar{\Psi}_{2k+1}$ have the gravitational scaling dimension k same as Φ_{2k+1} besides the logarithmic factor.

4. 2D type IIA superstring

The type II superstring theory discussed in Refs. [11, 12, 13] has the target space $(\varphi, x) \in$ (Liouville direction) $\times$ (S^1 with self-dual radius). The holomorphic energy-momentum tensor on the string world-sheet is

$$T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x\partial\psi_x - \frac{1}{2}(\partial\varphi)^2 + \frac{Q}{2}\partial^2\varphi - \frac{1}{2}\psi_\ell\partial\psi_\ell \tag{39}$$

with $Q = 2$, except ghosts' part. ψ_x and ψ_ℓ are superpartners of x and φ, respectively. Target-space supercurrents in the type IIA theory

$$q_+(z) = e^{-\frac{1}{2}\phi(z) - \frac{i}{2}H(z) - ix(z)}, \qquad \bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}(\bar{z}) + i\bar{x}(\bar{z})} \tag{40}$$

exist only for the S^1 target space of the self-dual radius. ϕ ($\bar{\phi}$) is the holomorphic (anti-holomorphic) bosonized superconformal ghost, and the fermions are bosonized as $\psi_\ell \pm i\psi_x = \sqrt{2}\, e^{\mp iH}$, $\bar{\psi}_\ell \pm i\bar{\psi}_x = \sqrt{2}\, e^{\mp i\bar{H}}$. Then the supercharges

$$Q_+ = \oint \frac{dz}{2\pi i}\, q_+(z), \qquad \bar{Q}_- = \oint \frac{d\bar{z}}{2\pi i}\, \bar{q}_-(\bar{z}) \tag{41}$$

are nilpotent $Q_+^2 = \bar{Q}_-^2 = 0$, which indeed matches the supercharges Q and $\bar{Q}$ in the matrix model.

The spectrum except special massive states is represented by the NS "tachyon" vertex operator (in (-1) picture):

$$T_k = e^{-\phi + ikx + p_\ell \varphi}, \qquad \bar{T}_{\bar{k}} = e^{-\bar{\phi} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}}, \tag{42}$$

and by the R vertex operator (in $(-\frac{1}{2})$ picture):

$$V_{k,\,\epsilon} = e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon H + ikx + p_\ell \varphi}, \qquad \bar{V}_{\bar{k},\,\bar{\epsilon}} = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{\epsilon}\bar{H} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}} \tag{43}$$

with $\epsilon, \bar{\epsilon} = \pm 1$. Locality with the supercurrents, mutual locality, superconformal invariance (including the Dirac equation constraint) and the level matching condition determine physical vertex operators. As discussed in [13], there are two consistent sets of physical vertex operators - "momentum background" and "winding background". Let us consider the "winding background". [2] The physical spectrum in the "winding background" is given by

$$\begin{aligned}
&(\mathrm{NS,\ NS}): && T_k\, \bar{T}_{-k} && (k \in \mathbf{Z} + \tfrac{1}{2}), \\
&(\mathrm{R+,\ R-}): && V_{k,\,+1}\, \bar{V}_{-k,\,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots), \\
&(\mathrm{R-,\ R+}): && V_{-k,\,-1}\, \bar{V}_{k,\,+1} && (k = 0, 1, 2, \cdots), \\
&(\mathrm{NS,\ R-}): && T_{-k}\, \bar{V}_{-k,\,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots), \\
&(\mathrm{R+,\ NS}): && V_{k,\,+1}\, \bar{T}_k && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),
\end{aligned} \tag{44}$$

where we take a branch of $p_\ell = 1 - |k|$ satisfying the locality bound $p_\ell \leq Q/2 = 1$ [18]. We can see that the vertex operators

$$V_{\frac{1}{2},+1}\, \bar{V}_{-\frac{1}{2},-1}, \qquad T_{-\frac{1}{2}}\, \bar{V}_{-\frac{1}{2},-1}, \qquad V_{\frac{1}{2},+1}\, \bar{T}_{\frac{1}{2}}, \qquad T_{-\frac{1}{2}}\, \bar{T}_{\frac{1}{2}} \tag{45}$$

[2] We can repeat the parallel argument for "momentum background" in the type IIB theory, which is equivalent to the "winding background" in the type IIA theory through T-duality with respect to the S^1 direction.

form a quartet under Q_+ and $\bar{Q}_-$: [3]

$$\begin{aligned} [Q_+, V_{\frac{1}{2},+1}\bar{V}_{-\frac{1}{2},-1}] &= T_{-\frac{1}{2}}\bar{V}_{-\frac{1}{2},-1}, \quad \{Q_+, T_{-\frac{1}{2}}\bar{V}_{-\frac{1}{2},-1}\} = 0, \\ \{Q_+, V_{\frac{1}{2},+1}\bar{T}_{\frac{1}{2}}\} &= T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}, \qquad [Q_+, T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}] = 0, \end{aligned} \tag{46}$$

$$\begin{aligned} [\bar{Q}_-, V_{\frac{1}{2},+1}\bar{V}_{-\frac{1}{2},-1}] &= -V_{\frac{1}{2},+1}\bar{T}_{\frac{1}{2}}, \quad \{\bar{Q}_-, V_{\frac{1}{2},+1}\bar{T}_{\frac{1}{2}}\} = 0, \\ \{\bar{Q}_-, T_{-\frac{1}{2}}\bar{V}_{-\frac{1}{2},-1}\} &= T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}, \qquad [\bar{Q}_-, T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}] = 0. \end{aligned} \tag{47}$$

Notice that (46) and (47) are isomorphic to (2) and (3), respectively. It leads to correspondence of single-trace operators in the matrix model to integrated vertex operators in the type IIA theory:

$$\begin{aligned} \Phi_1 &\iff \mathcal{V}_\phi(0) \equiv \int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}), \\ \Psi_1 &\iff \mathcal{V}_\psi(0) \equiv \int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}), \\ \bar{\Psi}_1 &\iff \mathcal{V}_{\bar{\psi}}(0) \equiv \int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}), \\ \frac{1}{N}\mathrm{tr}\,(-iB) &\iff \mathcal{V}_B(0) \equiv \int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}), \end{aligned} \tag{48}$$

which is consistent with the identification in (9)–(12). Furthermore, it is natural to extend (48) to case of higher $k(=1,2,\cdots)$ as

$$\begin{aligned} \Phi_{2k+1} &\iff \mathcal{V}_\phi(k) \equiv \int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}), \\ \Psi_{2k+1} &\iff \mathcal{V}_\psi(k) \equiv \int d^2z\, T_{-k-\frac{1}{2}}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}), \\ \bar{\Psi}_{2k+1} &\iff \mathcal{V}_{\bar{\psi}}(k) \equiv \int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{T}_{k+\frac{1}{2}}(\bar{z}). \end{aligned} \tag{49}$$

Since the "tachyons" of the negative winding $\int d^2z\, T_{-k-\frac{1}{2}}(z)\,\bar{T}_{k+\frac{1}{2}}(\bar{z})$ $(k = 0,1,2,\cdots)$ are invariant under Q_+ and $\bar{Q}_-$, they are expected to be mapped to $\{\frac{1}{N}\mathrm{tr}\,(-iB)^{k+1}\}$ $(k = 0,1,2,\cdots)$ perhaps with some mixing terms. We see in (49) that the powers of matrices are interpreted as windings or momenta in the S^1 direction of the type IIA theory. Although such interpretation is not usual in matrix models for two-dimensional quantum gravity coupled to $c < 1$ matters, refs. [19] show that a positive power k of a matrix variable in the Penner model correctly describe the "tachyons" with negative momentum $-k$ in the $c = 1$ string on S^1, which is in harmony with our

[3] We here assume that Q_+ commutes with $\bar{T}_{\bar{k}}$ and anti-commutes with $\bar{V}_{\bar{k},\bar{\epsilon}}$, and that $\bar{Q}_-$ commutes with T_k and anti-commutes with $V_{k,\epsilon}$. It is plausible from the statistics in the target space. In ref. [2], we introduce cocycle factors to the vertex operators in order to realize the (anti-)commutation properties.

interpretation. In [19], the positive momentum "tachyons" are represented by introducing source terms of an external matrix via the Kontsevich-Miwa transformation in the Penner model. In turn, it is natural to expect in our case that the positive winding "tachyons" $\int d^2z\, T_{-k-\frac{1}{2}}(z)\, \bar{T}_{k+\frac{1}{2}}(\bar{z})$ $(k=-1,-2,\cdots)$ in the type IIA theory are expressed in a similar manner in the matrix model.

Note that (R−, R+) operators are singlets under the target-space SUSYs $Q_+, \bar{Q}_-$, and appear to have no counterpart in the matrix model side. Since the expectation value of operators carrying the nonzero RR charge $\langle \Phi_{2k+1}\rangle_0$ does not vanish as seen in (14), the matrix model is considered to correspond to the type IIA theory on a background of the (R−, R+) fields. We may introduce the (R−, R+) background in the form of vertex operators, when the strength of the background $(\nu_+ - \nu_-)$ is small.

5. Correspondence between the matrix model and the type IIA theory

Correlation functions among integrated vertex operators in the type IIA theory on the trivial background are given by

$$\left\langle \prod_i \mathcal{V}_i \right\rangle = \frac{1}{\mathrm{Vol.(CKV)}} \int \mathcal{D}(x,\varphi,H,\mathrm{ghosts})\, e^{-S_{\mathrm{CFT}}} e^{-S_{\mathrm{int}}} \prod_i \mathcal{V}_i, \qquad (50)$$

where

$$\begin{aligned} S_{\mathrm{CFT}} &= \frac{1}{2\pi}\int d^2z \left[\partial x \bar{\partial} x + \partial\varphi\bar{\partial}\varphi + \frac{Q}{4}\sqrt{\hat{g}}\hat{R}\varphi + \partial H\bar{\partial}H + (\mathrm{ghosts})\right], \\ S_{\mathrm{int}} &= \mu_1 \int d^2z\, T^{(0)}_{-\frac{1}{2}}(z)\bar{T}^{(0)}_{\frac{1}{2}}(\bar{z}). \end{aligned} \qquad (51)$$

The 0-picture (NS, NS) "tachyon" is given by

$$\begin{aligned} T^{(0)}_k &= \frac{i}{\sqrt{2}}\left[(p_\ell - k)e^{iH} + (p_\ell + k)e^{-iH}\right] e^{ikx + p_\ell\varphi}, \\ \bar{T}^{(0)}_{\bar{k}} &= \frac{i}{\sqrt{2}}\left[(p_\ell - \bar{k})e^{i\bar{H}} + (p_\ell + \bar{k})e^{-i\bar{H}}\right] e^{i\bar{k}\bar{x} + p_\ell\bar{\varphi}}. \end{aligned} \qquad (52)$$

We consider correlation functions in the IIA theory on a nontrivial (R−, R+) background as a form

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i\right) e^{W_{\mathrm{RR}}} \right\rangle. \qquad (53)$$

The background W_{RR} is invariant under the target-space SUSYs:

$$W_{\mathrm{RR}} = (\nu_+ - \nu_-) \sum_{k\in\mathbf{Z}} a_k\, \mu_1^{k+1} \mathcal{V}^{\mathrm{RR}}_k,$$

$$\mathcal{V}_k^{\rm RR} \equiv \begin{cases} \int d^2z\, V_{k,-1}(z)\bar{V}_{-k,+1}(\bar{z}) & (p_\ell = 1-|k|,\, k \le 0) \\ \int d^2z\, V_{-k,-1}^{(\rm nonlocal)}(z)\bar{V}_{k,+1}^{(\rm nonlocal)}(\bar{z}) & (p_\ell = 1+|k|,\, k \ge 1). \end{cases} \tag{54}$$

a_k is a numerical constant. Although the nonlocal operators in (54) with $p_\ell > 1$ do not satisfy the Dirac equation constraint on the trivial background, these operators are necessary to see the correspondence to the matrix model as we see later. Since the RR background possibly change the on-shell condition, it seems not so strange. We treat the RR background for $(\nu_+ - \nu_-)$ small as

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i\right) e^{W_{\rm RR}} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\prod_i \mathcal{V}_i\right) (W_{\rm RR})^n \right\rangle, \tag{55}$$

and the picture is adjusted by hand so that the total picture is equal to -2.

In computation of amplitudes in the type IIA theory, we consider the so-called $s = 0$ amplitude in the Liouville theory, which is interpreted as a bulk amplitude insensitive to details of the Liouville wall [20]. It is considered to be in harmony with considering the leading nontrivial contribution for small $(\nu_+ - \nu_-)$, because higher orders of $(\nu_+ - \nu_-)$ seems to detect a cigar geometry deformed from the two-dimensional target space (Liouville direction) $\times$ (S^1 with self-dual radius) [12]. The direction to the Liouville wall corresponds to the direction to the tip of the cigar. The Liouville computation yields

$$\left\langle \mathcal{V}_B(0)\, \mathcal{V}_\phi(k)\, \mathcal{V}_\ell^{\rm RR} \right\rangle = \delta_{k,\ell}\,(2\ln\mu_1), \tag{56}$$

$$\left\langle \mathcal{V}_\phi(k_1), \mathcal{V}_\phi(k_2)\, \mathcal{V}_{\ell_1}^{\rm RR}\, \mathcal{V}_{\ell_2}^{\rm RR} \right\rangle = (\delta_{\ell_1,k_1+k_2}\delta_{\ell_2,-1} + (\ell_1 \leftrightarrow \ell_2)) \times \frac{\pi}{2} \left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 c_L (2\ln\mu_1)^2. \tag{57}$$

In the computation (57), we encountered the integral

$$\int d^2z\, z^\alpha \bar{z}^{\bar{\alpha}} (1-z)^\beta (1-\bar{z})^{\bar{\beta}} = \pi \frac{\Gamma(\bar{\alpha}+1)\Gamma(\bar{\beta}+1)}{\Gamma(\bar{\alpha}+\bar{\beta}+2)} \frac{\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \tag{58}$$

with

$$\alpha = \bar{\alpha} = k_1 + k_2, \qquad \beta = \bar{\beta} = -k_1 - 1, \qquad (k_1, k_2 = 0, 1, 2, \cdots). \tag{59}$$

This expression is indefinite. We computed it by regularizing as

$$\alpha \to \alpha + \epsilon, \qquad \bar{\alpha} \to \bar{\alpha} + \epsilon, \qquad \beta \to \beta + \epsilon, \qquad \bar{\beta} \to \bar{\beta} + \epsilon, \tag{60}$$

where $\epsilon = \frac{1}{c_L V_L}$. $V_L \equiv 2\ln\frac{1}{\mu_1}$ is the Liouville volume, and c_L is a numerical constant.

Let us identify the coupling μ_1 of the Liouville interaction $S_{\rm int}$ in (51) with the "cosmological constant" ω by shifting the origin of the Liouville coordinate. Then, it leads to the identification

$$N\,{\rm tr}(-iB) \cong \frac{1}{4}\int d^2z\, T^{(0)}_{-\frac{1}{2}}(z)\,\bar{T}^{(0)}_{\frac{1}{2}}(\bar{z}). \tag{61}$$

Also, introducing coefficients c_k, d_k, $\bar{d}_k$, we precisely express the correspondence in (48) and (49) as

$$\Phi_{2k+1} \cong c_k \mathcal{V}_\phi(k), \qquad \Psi_{2k+1} \cong d_k \mathcal{V}_\psi(k), \qquad \bar{\Psi}_{2k+1} \cong \bar{d}_k \mathcal{V}_{\bar{\psi}}(k). \tag{62}$$

We put the overall normalization factor $\mathcal{N}$ in identifying the amplitudes in the matrix-model side and those in the IIA theory side:

$$\langle N{\rm tr}(-iB)\,\Phi_{2k+1}\rangle_{C,0} \cong \mathcal{N}\left\langle\!\!\left\langle\, \frac{1}{4}\left(\int T^{(0)}_{-\frac{1}{2}}\bar{T}^{(0)}_{\frac{1}{2}}\right)\, c_k\mathcal{V}_\phi(k)\,\right\rangle\!\!\right\rangle. \tag{63}$$

The left hand side is calculated by using (14):

$$\begin{aligned}
({\rm LHS}) &= -\frac{1}{4}\partial_\omega\,\langle\Phi_{2k+1}\rangle_0 \\
&\sim -(\nu_+-\nu_-)\frac{2^k}{\pi}\frac{(2k+1)!!}{(k+1)!}\,\omega^{k+1}\ln\omega.
\end{aligned} \tag{64}$$

On the other hand, under a suitable choice of the picture, leading nontrivial contribution for $(\nu_+-\nu_-)$ small to the right hand side is

$$\begin{aligned}
&\frac{1}{4}\mathcal{N}c_k\,\langle\mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,W_{\rm RR}\rangle \\
&\qquad = \frac{1}{4}\mathcal{N}c_k(\nu_+-\nu_-)\sum_{\ell\in\mathbf{Z}} a_\ell\,\omega^{\ell+1}\left\langle\mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,\mathcal{V}^{\rm RR}_\ell\right\rangle \\
&\qquad = \frac{1}{2}(\nu_+-\nu_-)\mathcal{N}c_k\,a_k\,\omega^{k+1}\ln\omega,
\end{aligned} \tag{65}$$

where (56) was used. So, the identification (63) leads to

$$\mathcal{N}c_k\,a_k = -\frac{2}{\pi}\frac{(2k+1)!}{k!(k+1)!}. \tag{66}$$

Next, let us consider the correspondence

$$\langle\Phi_{2k_1+1}\Phi_{2k_2+1}\rangle_{C,0} \cong \mathcal{N}\,\langle\!\langle\, c_{k_1}\mathcal{V}_\phi(k_1)\,c_{k_2}\mathcal{V}_\phi(k_2)\,\rangle\!\rangle\,. \tag{67}$$

Leading nontrivial contribution to the right hand side is obtained from (57) as

$$
\begin{aligned}
&\mathcal{N} c_{k_1}\, c_{k_2} \left\langle \mathcal{V}_\phi(k_1)\, \mathcal{V}_\phi(k_2)\, \frac{1}{2!}(W_{\mathrm{RR}})^2 \right\rangle \\
&\quad = \frac{1}{2}\mathcal{N} c_{k_1}\, c_{k_2}\, (\nu_+ - \nu_-)^2 \sum_{\ell_1,\ell_2\in\mathbf{Z}} a_{\ell_1}\, a_{\ell_2}\, \omega^{\ell_1+\ell_2+2} \left\langle \mathcal{V}_\phi(k_1)\, \mathcal{V}_\phi(k_2)\, \mathcal{V}^{\mathrm{RR}}_{\ell_1}\, \mathcal{V}^{\mathrm{RR}}_{\ell_2} \right\rangle \\
&\quad = (\nu_+ - \nu_-)^2 \mathcal{N} c_L\, c_{k_1}\, c_{k_2}\, a_{k_1+k_2}\, a_{-1}\, 2\pi \left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 \omega^{k_1+k_2+1}\, (\ln\omega)^2.
\end{aligned} \tag{68}
$$

The result of the left hand side is given by (29). Comparing these we have

$$
\begin{aligned}
&\left(\frac{c_{k_1}}{(2k_1+1)!}\right)\left(\frac{c_{k_2}}{(2k_2+1)!}\right)(a_{k_1+k_2}(k_1+k_2)!(k_1+k_2+1)!) \\
&\qquad\qquad = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L a_{-1}}.
\end{aligned} \tag{69}
$$

It is solved as

$$
c_k = c_0(2k+1)!, \qquad a_k = \frac{a_0}{k!(k+1)!}\,, \tag{70}
$$

for $k = 0, 1, 2, \cdots$ with

$$
c_0^2 a_0 = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L\, a_{-1}}\,. \tag{71}
$$

Remarkably, (66) is consistent to (70). It serves a quite nontrivial check of the correspondence.

Also, the correspondence of the amplitudes containing fermions

$$
\begin{aligned}
\langle \Psi_1 \bar{\Psi}_1 \rangle_{C,0} &\cong \mathcal{N} \left\langle\!\!\left\langle\, d_0 \mathcal{V}_\psi(0)\, \bar{d}_0 \mathcal{V}_{\bar{\psi}}(0) \,\right\rangle\!\!\right\rangle, \\
\langle \Psi_3 \bar{\Psi}_3 \rangle_{C,0} &\cong \mathcal{N} \left\langle\!\!\left\langle\, d_1 \mathcal{V}_\psi(1)\, \bar{d}_1 \mathcal{V}_{\bar{\psi}}(1) \,\right\rangle\!\!\right\rangle
\end{aligned} \tag{72}
$$

yields

$$
d_0 \bar{d}_0 = -\frac{1}{4} c_0, \qquad d_1 \bar{d}_1 = \frac{3}{\pi^2}\frac{c_0}{a_0^2}. \tag{73}
$$

It leads to the precise correspondence between the supercharges:

$$
Q \cong \frac{d_0}{c_0}\, Q_+, \qquad \bar{Q} \cong \frac{\bar{d}_0}{c_0}\, \bar{Q}_-. \tag{74}
$$

So far, the correspondence seems consistent at the level of planar or tree amplitudes.

6. Summary and Discussion

We computed planar correlation functions in the double-well SUSY matrix model, and discussed its correspondence to 2D type IIA superstring theory on (R−,R+) background by comparing amplitudes in both sides. This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

It is interesting to examine the correspondence at deeper level in higher genus amplitudes and in amplitudes containing special massive operators. Also, it is important to discuss the correspondence in the off-shell formulation such as the hybrid formalism [14].

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