Noncommutative Deformations of $\mathbb{C}P^N$ and $\mathbb{C}H^N$*

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Abstract

We give explicit formulas for star products in $\mathbb{C}P^N$ and $\mathbb{C}H^N$ in all order of a noncommutative parameter. We use a quantization method to perform a deformation quantization of Kähler manifolds, which is introduced by Karabegov. We also investigate the Fock representations of the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$.

1. Introduction

Noncommutative spaces appear in various physical theories (For review, see e.g. [1, 2]). For example, let us consider a charged particle in an $x-y$ plane in a strong magnetic field perpendicular to the plane. The Lagrangian of the particle is dominated by an interaction term, $B_z(\dot{x}(t)y(t) - \dot{y}(t)x(t))$, and then the coordinates become noncommutative, $[x(t),\ y(t)] \sim i\hbar/B_z$, after the canonical quantization. In string theories, a similar phenomenon occurs in low energy effective theories on D-branes in a constant background NS-NS B field. In this case, effective theories on D-branes become nonabelian gauge theories on noncommutative D-branes with a noncommutative parameter characterized by a value of B.

An important example of noncommutative spaces which are frequently used in investigations of field theories is the noncommutative $\mathbb{R}^d$. A product between fields on the noncommutative $\mathbb{R}^d$ is given by the Moyal product,

$$(f * g)(x) = e^{\frac{i}{2}\theta^{ij}\partial_i^x\partial_j^y} f(x)g(y)\Big|_{y=x},$$

where θ^{ij} is a constant noncommutative parameter, $\theta^{ij} = -\theta^{ji}$. In particular, the coordinates satisfy the following commutation relations under the Moyal product,

$$[x^i,\ x^j]_* = x^i * x^j - x^j * x^i = i\theta^{ij}.$$

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These are the same forms as the commutation relations which appear in a system of a charged particle in a strong magnetic field mentioned above.

Matrix models give other important examples of noncommutative spaces and field theories on them. In matrix models, all of degrees of freedom are described by finite size matrices. Let us consider the fuzzy S^2. The coordinates on the fuzzy S^2, $\hat{X}_i$ $(i = 1, 2, 3)$, are described by using the $(n+1)$-dimensional representation of $su(2)$ algebra $(n \in \mathbb{N})$,

$$[\hat{X}_i, \hat{X}_j] = i\epsilon_{ijk}\hat{X}_k.$$

Fields on the fuzzy S^2 are given by matrices which are functions of $\hat{X}_i$, and thus the product between fields are noncommutative obviously.

Through vigorous investigations of field theories on noncommutative spaces, it is clarified that those theories have some characteristic properties. One of the properties is nonlocality. A typical example is known as UV/IR mixing, which is the fact that low energy scales appear in high energy phenomena (e.g. UV divergences). Moreover, in cases of gauge theories, open Wilson lines become gauge invariant observables and their lengths are proportional to their momenta. Another property is the existence of noncommutative solitons and instantons. In most cases, these properties are studied in field theories on a limited class of noncommutative spaces, e.g. the noncommutative $\mathbb{R}^d$, the fuzzy tori, the fuzzy spheres, and so on. To investigate these properties further, one need to construct a wide class of noncommutative spaces in which physical quantities can be calculated explicitly.

In this article, we give explicit formulas of a deformation quantization with separation of variables for $\mathbb{C}P^N$ and $\mathbb{C}H^N$. Star products are obtained as power series of the noncommutative parameter in all order. We also give the Fock representations of the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$ [3].

2. Review of the deformation quantization with separation of variables

In this section, we review the deformation quantization with separation of variables to construct noncommutative Kähler manifolds.

An N-dimensional complex Kähler manifolds is defined by using a Kähler potential. Let Φ be a Kähler potential and ω be a Kähler 2-form:

$$\begin{aligned} \omega &:= i g_{k\bar{l}} dz^k \wedge d\bar{z}^l, \\ g_{k\bar{l}} &:= \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}\,. \end{aligned} \tag{1}$$

The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$:

$$g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}. \tag{2}$$

In the following, we denote

$$\partial_k = \frac{\partial}{\partial z^k}, \qquad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}\,. \tag{3}$$

Firstly, we give the definition of deformation quantization as follows. $\mathcal{F}$ is defined as a set of formal power series:

$$\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k,\ f_k \in C^\infty \right\}. \tag{4}$$

A star product is defined as

$$f * g = \sum_k C_k(f,g)\hbar^k \tag{5}$$

such that the product satisfies the following conditions.

1. $*$ is associative product.
2. C_k is a bidifferential operator.
3. C_0 and C_1 are defined as

$$C_0(f,g) = fg, \tag{6}$$
$$C_1(f,g) - C_1(g,f) = i\{f,g\}, \tag{7}$$

   where $\{f,g\}$ is the Poisson bracket.
4. $f * 1 = 1 * f = f$.

Note that this definition of the deformation quantization is weaker than the usual definition of deformation quantization. The difference between them is in (7). In the strong sense of deformation quantization the condition $C_1(f,g) = \frac{i}{2}\{f,g\}$ is required. Deformation quantization with the separation of variables does not satisfy this condition. In this article, "deformation quantization" is used in this weak sense.

Next, a star product with separation of variables is defined as follows. $*$ is called a star product with separation of variables when

$$a * f = af \tag{8}$$

for a holomorphic function a and

$$f * b = fb \tag{9}$$

for an anti-holomorphic function b.

Karabegov proposed a method to construct a star product with separation of variables on Kähler manifolds[4, 5, 6]. In his method, a star product by

$f \in \mathcal{F}$ is given as a differential operator L_f such that $f * g = L_f g$. L_f is expanded as a power series of $\hbar$,

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n, \tag{10}$$

where A_n is a differential operator whose coefficients depend on f i.e.

$$A_n = a_{n,\alpha}(f) D^{\alpha}, \; D^{\alpha} = \prod_{i=1}^{n} (D^{\bar{i}})^{\alpha_i}, \; (D^{\bar{i}}) = g^{\bar{i}l} \partial_l. \tag{11}$$

Here α is a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$. L_f is uniquely determined by the following conditions,

$$[L_f, \partial_{\bar{l}} \Phi + \hbar \partial_{\bar{l}}] = 0, \tag{12}$$

$$L_f 1 = f * 1 = f. \tag{13}$$

Then, it can be shown that this $*$-product satisfies the associativity, $L_h(L_g f) = h * (g * f) = (h * g) * f = L_{L_h g} f$.

When the operator $L_{\bar{z}^i}$ corresponding to the left-multiplication by $\bar{z}^i$ is constructed, a left operation L_f for a generic function f is given by the following formula,

$$L_f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \bar{z}} \right)^{\alpha} f \; (L_{\bar{z}} - \bar{z})^{\alpha}. \tag{14}$$

3. Star product with separation of variables on $\mathbb{C}\mathrm{P}^{\mathrm{N}}$

In this section, we give explicit expressions for a star product on $\mathbb{C}P^N$ by solving the condition (12).

In the inhomogeneous coordinates z^i $(i = 1, 2, \cdots, N)$, the Kähler potential of $\mathbb{C}P^N$ is given by

$$\Phi = \ln \left(1 + |z|^2 \right), \tag{15}$$

where $|z|^2 = \sum_{k=1}^{N} z^k \bar{z}^k$. The metric $(g_{i\bar{j}})$ is

$$ds^2 = 2 g_{i\bar{j}} dz^i d\bar{z}^j, \tag{16}$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}, \tag{17}$$

and the inverse of the metric $(g^{\bar{i}j})$ is

$$g^{\bar{i}j} = (1 + |z|^2) \left(\delta_{ij} + z^j \bar{z}^i \right). \tag{18}$$

In the case of $\mathbb{C}P^N$, the following relations simplify our calculations for L_f,

$$\partial_{\bar{i}_1}\partial_{\bar{i}_2}\cdots\partial_{\bar{i}_n}\Phi = (-1)^{n-1}(n-1)!\ \partial_{\bar{i}_1}\Phi\partial_{\bar{i}_2}\Phi\cdots\partial_{\bar{i}_n}\Phi, \tag{19}$$

$$R_{i\bar{j}k\bar{l}} = -\, g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}}, \tag{20}$$

where $R_{i\bar{j}k\bar{l}}$ is the Riemann tensor.

We construct the operator $L_{\bar{z}^l}$, which is corresponding to the left star product by $\bar{z}^l$. $L_{\bar{z}^l}$ is defined as a power series of $\hbar$,

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty}\hbar^n A_n, \tag{21}$$

where A_n $(n \geq 2)$ is a formal series of the differential operators $D^{\bar{k}}$. We assume that A_n has the following form,

$$A_n = \sum_{m=2}^{n} a_m^{(n)}\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}}, \tag{22}$$

where the coefficients $a_m^{(n)}$ do not depend on the coordinates..

From the requirement of $[L_{\bar{z}^l}, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0$, the operators A_n are recursively determined by the equations

$$[A_n, \partial_{\bar{i}}\Phi] = [\partial_{\bar{i}}, A_{n-1}]\,, \qquad (n \geq 2) \tag{23}$$

where $A_1 = D^{\bar{l}}$. $A_2 = \partial_{\bar{j}}\Phi D^{\bar{j}}D^{\bar{l}}$ is easily obtained from the above equation. Substituting the assumption (22) into the recursion relations, we find

$$a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1, \tag{24}$$

and

$$a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1)a_m^{(n-1)}. \tag{25}$$

To solve this equation, we introduce a generating function

$$\alpha_m(t) \equiv \sum_{n=m}^{\infty} t^n a_m^{(n)}, \tag{26}$$

for $m \geq 2$. Then the relation (25) is written as

$$\alpha_m(t) = t\left[\alpha_{m-1}(t) + (m-1)\alpha_m(t)\right], \tag{27}$$

and can be solved as

$$\begin{aligned} \alpha_m(t) &= \frac{t}{1-(m-1)t}\alpha_{m-1}(t) \\ &= t^{m-2}\prod_{n=2}^{m-1}\frac{1}{1-nt}\times\alpha_2(t). \end{aligned} \tag{28}$$

From (24), $\alpha_2(t)$ becomes

$$\alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1-t}, \tag{29}$$

and thus $\alpha_m(t)$ is determined as

$$\alpha_m(t) = t^m \prod_{n=1}^{m-1}\frac{1}{1-nt} = \frac{\Gamma(1-m+\frac{1}{t})}{\Gamma(1+\frac{1}{t})}, \qquad (m \geq 2). \tag{30}$$

Actually, $a_m^{(n)}$ is related to the Stirling numbers of the second kind $S(n,k)$ as

$$a_m^{(n)} = S(n-1, m-1). \tag{31}$$

Summarizing the above calculations, $L_{\bar{z}^l}$ becomes

$$\begin{aligned} L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty}\hbar^n\sum_{m=2}^{n} a_m^{(n)}\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}} \\ &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty}\hbar^n a_m^{(n)}\right)\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}} \\ &= \bar{z}^l + \sum_{m=1}^{\infty}\alpha_m(\hbar)\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}}. \end{aligned} \tag{32}$$

Here we defined $\alpha_1(t) = t$. Similarly, it can be shown that the right star product by z^l, $R_{z^l}f = f * z^l$ is expressed as

$$\begin{aligned} R_{z^l} &= z^l + \hbar D^l + \sum_{n=2}^{\infty}\hbar^n\sum_{m=2}^{n} a_m^{(n)}\partial_{j_1}\Phi\cdots\partial_{j_{m-1}}\Phi D^{j_1}\cdots D^{j_{m-1}}D^l \\ &= z^l + \sum_{m=1}^{\infty}\alpha_m(\hbar)\partial_{j_1}\Phi\cdots\partial_{j_{m-1}}\Phi D^{j_1}\cdots D^{j_{m-1}}D^l, \end{aligned} \tag{33}$$

where $D^i = g^{i\bar{j}}\partial_{\bar{j}}$.

From above formulas, we can now calculate the star products among z^i and $\bar{z}^i$, explicitly,

$$z^i * z^j = z^i z^j, \tag{34}$$

$$z^i * \bar{z}^j = z^i \bar{z}^j, \tag{35}$$

$$\bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \tag{36}$$

$$\begin{aligned} \bar{z}^i * z^j = {} & \bar{z}^i z^j + \hbar\delta_{ij}(1+|z|^2)\,{}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right) \\ & + \frac{\hbar}{1-\hbar}\bar{z}^i z^j (1+|z|^2)\,{}_2F_1\left(1,2;2-1/\hbar;-|z|^2\right), \end{aligned} \tag{37}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Though the differential operator L_f corresponding to the left multiplication by a generic function f can be derived from the formula (14), we find that it can be also written as the following form,

$$L_f = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) D^{\bar{k}_1} \cdots D^{\bar{k}_n}. \tag{38}$$

The coefficient $c_n(\hbar)$ is determined by the condition $[L_f, \hbar\partial_{\bar{i}} + \partial_{\bar{i}}\Phi] = 0$. This condition leads to the recurrence relation, $n(1-\hbar(n-1))c_n(\hbar) - \hbar c_{n-1}(\hbar) = 0$. This equation is easily solved and $c_n(\hbar)$ is obtained as

$$c_n(\hbar) = \frac{\Gamma(1-n+1/\hbar)}{n!\Gamma(1+1/\hbar)} = \frac{\alpha_n(\hbar)}{n!}, \tag{39}$$

under the initial condition $c_0 = 1$. Furthermore, we can show that the expression of L_f (38) is rewritten by the use of the covariant derivatives on the manifolds, as follows,

$$L_f g = f * g = \sum_{n=0}^{\infty} c_n(\hbar) g^{\bar{j}_1 k_1} \cdots g^{\bar{j}_n k_n} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f\right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g\right). \tag{40}$$

Here we used the fact that non-vanishing components of the Christoffel symbols on a Kähler manifolds are only Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$.

In this article, we treat $\hbar$ as a formal parameter. Here we consider the specific case of $\hbar = 1/L$ ($L \in \mathbb{N}$) and the star product in a function space $\mathcal{M}_L$ spanned by

$$\frac{z^{i_1} \dots z^{i_m} \bar{z}^{j_1} \dots \bar{z}^{j_n}}{(1+|z|^2)^L}, \qquad (m, n \le L).$$

In this case, the series in (38) terminates at $n = L$, because

$$D^{j_1} \cdots D^{j_{L+1}} f = 0, \qquad D^{\bar{k}_1} \cdots D^{\bar{k}_{L+1}} g = 0,$$

where $f, g \in \mathcal{M}_L$. Then, the expression of the star product coincides with the one in [7].

Several ways of making deformation quantization by a reduction from higher dimensional manifolds are known. In [8], a star product on $\mathbb{C}P^N$ was constructed by performing the phase space reduction from $\mathbb{C}^{N+1}\backslash\{0\}$. The expression of their star product, denoted here as $*_B$, for functions f and g on $\mathbb{C}P^N$ is written as

$$\begin{aligned} & f *_B g \\ & = fg + \sum_{m=1}^{\infty} \hbar^m \sum_{s=1}^{m} \sum_{k=1}^{s} \frac{k^{m-1}(-1)^{m-k}}{s!(s-k)!(k-1)!} \left(|\zeta|^2\right)^s \frac{\partial^s f}{\partial\bar{\zeta}^{A_1}\cdots\bar{\zeta}^{A_s}} \frac{\partial^s g}{\partial\zeta^{A_1}\cdots\zeta^{A_s}}, \end{aligned} \tag{41}$$

where $\zeta^{A_i}, \bar{\zeta}^{A_j}$ are the homogeneous coordinates. This satisfies the conditions for a star product with separation of variables, and thus the equations (34)-(36) hold trivially under $*_B$ product. A nontrivial one is $\bar{z}^i *_B z^j$, and it is calculated as

$$\bar{z}^i *_B z^j = \bar{z}^i z^j + \hbar\delta_{ij}(1+|z|^2)\tilde{F}_1(-|z|^2) + \hbar\bar{z}^i z^j(1+|z|^2)\tilde{F}_2(-|z|^2), \tag{42}$$

where $z^i = \zeta^i/\zeta^0, \bar{z}^i = \bar{\zeta}^i/\bar{\zeta}^0$, and

$$\tilde{F}_1(-|z|^2) \equiv \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m s! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!}(1+|z|^2)^s, \tag{43}$$

$$\tilde{F}_2(-|z|^2) \equiv \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m (s+1)! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!}(1+|z|^2)^s. \tag{44}$$

We can show that $\tilde{F}_1(-|z|^2)$ satisfies the hypergeometric equation and the boundary conditions for ${}_2F_1(1,1;1-1/\hbar;-|z|^2)$, and thus $\tilde{F}_1(-|z|^2) = {}_2F_1(1,1;1-1/\hbar;-|z|^2)$. Similarly, $\tilde{F}_2(-|z|^2) = {}_2F_1(1,2;2-1/\hbar;-|z|^2)/(1-\hbar)$ can be also shown. Therefore it turns out $\bar{z}^i * z^j = \bar{z}^i *_B z^j$. These facts lead to $f * g = f *_B g$, since star products between generic functions are calculated by using these relations. Namely, the star product constructed by Karabegov's method coincides with the star product $*_B$ in [8]. As far as we know, the origin of this coincidence of the star products obtained by these different methods is not apparent at this time.

Before closing this section, we mention the Leibniz rule for differentials under the star product obtained here. Since the expression of the star product we obtained depends on the metric of $\mathbb{C}P^N$, the partial differentials does not satisfy the Leibniz rule, $\partial(f * g) \neq (\partial f) * g + f * (\partial g)$. However we can show that the Leibniz rule holds with respect to the Killing vector fields corresponding to the $SU(N+1)$ isometry of $\mathbb{C}P^N$,

$$\mathcal{L}_a(f * g) = (\mathcal{L}_a f) * g + f * (\mathcal{L}_a g). \tag{45}$$

Here $\mathcal{L}_a$ is the Killing vector fields,

$$[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c, \tag{46}$$

where f_{abc} is the structure constant of $su(N+1)$. This property is important to construct actions of field theories on the noncommutative $\mathbb{C}P^N$ which is invariant under the isometry.

4. Fock representation

The left star product by $\partial_i \Phi$ and the right star product by $\partial_{\bar{i}} \Phi$ are respectively written as

$$L_{\partial_i \Phi} = \hbar \partial_i + \partial_i \Phi = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar}, \tag{47}$$

$$R_{\partial_{\bar{i}} \Phi} = \hbar \partial_{\bar{i}} + \partial_{\bar{i}} \Phi = \hbar e^{-\Phi/\hbar} \partial_{\bar{i}} e^{\Phi/\hbar}. \tag{48}$$

From the definition of the star product given in the previous section, we easily find

$$\partial_i \Phi * z^j - z^j * \partial_i \Phi = \hbar \delta_{ij}, \quad z^i * z^j - z^j * z^i = \partial_i \Phi * \partial_j \Phi - \partial_j \Phi * \partial_i \Phi = 0, \tag{49}$$

$$\bar{z}^i * \partial_{\bar{j}} \Phi - \partial_{\bar{j}} \Phi * \bar{z}^i = \hbar \delta_{ij}, \quad \bar{z}^i * \bar{z}^j - \bar{z}^j * \bar{z}^i = \partial_{\bar{i}} \Phi * \partial_{\bar{j}} \Phi - \partial_{\bar{j}} \Phi * \partial_{\bar{i}} \Phi = 0. \tag{50}$$

Hence, $\{z^i, \partial_j \Phi \mid i,j = 1,2,\cdots,N\}$ and $\{\bar{z}^i, \partial_{\bar{j}} \Phi \mid i,j = 1,2,\cdots,N\}$ constitute $2N$ sets of the creation-annihilation operators under the star product. However, it is noted that operators in $\{z^i, \partial_j \Phi\}$ does not commute with ones in $\{\bar{z}^i, \partial_{\bar{j}} \Phi\}$, e.g., $z^i * \bar{z}^j - \bar{z}^j * z^i \neq 0$.

Here, we would like to construct the Fock representation of the star product. First we show that $e^{-\Phi/\hbar} = (1+|z|^2)^{-1/\hbar}$ corresponds to the vacuum projection under the star product. $e^{-\Phi/\hbar}$ is annihilated by the left star product of $\partial_i \Phi$ and $\bar{z}^i$,

$$\partial_i \Phi * e^{-\Phi/\hbar} = L_{\partial_i \Phi} e^{-\Phi/\hbar} = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar} e^{-\Phi/\hbar} = 0, \tag{51}$$

$$\begin{aligned} \bar{z}^i * e^{-\Phi/\hbar} &= L_{\bar{z}^i} e^{-\Phi/\hbar} \\ &= \left(\bar{z}^i + \sum_{m=1}^{\infty} \alpha_m(\hbar) \partial_{\bar{j}_1} \Phi \cdots \partial_{\bar{j}_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} \right) e^{-\Phi/\hbar} \\ &= 0. \end{aligned} \tag{52}$$

Similarly, it is shown that $e^{-\Phi/\hbar}$ is annihilated by the right star product of the $\partial_{\bar{i}} \Phi$ and z^i,

$$e^{-\Phi/\hbar} * \partial_{\bar{i}} \Phi = e^{-\Phi/\hbar} * z^i = 0. \tag{53}$$

Next, we show that $e^{-\Phi/\hbar}$ satisfies the relation

$$e^{-\Phi/\hbar} * f(z,\bar{z}) = e^{-\Phi/\hbar} f(0,\bar{z}) \tag{54}$$

for a function $f(z,\bar{z})$ such that $f(z,\bar{w})$ can be expanded as Taylor series with respect to z^i and $\bar{w}^j$, respectively. To show the relation, we note that the differential operator R_{z^i} corresponding to the right product of z^i contains only partial derivatives by $\bar{z}^j$, and thus commutes with z^k. Moreover, R_{z^i} annihilates $e^{-\Phi/\hbar}$, $R_{z^i} e^{-\Phi/\hbar} = e^{-\Phi/\hbar} * z^i = 0$ as mentioned above. From these, the relation (54) is shown as

$$\begin{aligned}
e^{-\Phi/\hbar} * f(z,\bar{z}) &= R_f e^{-\Phi/\hbar} \\
&= \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{1}{k_1!\cdots k_N!} \partial_1^{k_1}\cdots\partial_N^{k_N} f(z,\bar{z}) \prod_{m=1}^{N} (R_{z^m} - z^m)^{k_m} e^{-\Phi/\hbar} \\
&= \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{1}{k_1!\cdots k_N!} \partial_1^{k_1}\cdots\partial_N^{k_N} f(z,\bar{z}) \prod_{m=1}^{N} (-z^m)^{k_m} e^{-\Phi/\hbar} \\
&= e^{-\Phi/\hbar} f(0,\bar{z}).
\end{aligned} \tag{55}$$

Similarly, the following equation holds

$$f(z,\bar{z}) * e^{-\Phi/\hbar} = f(z,0) e^{-\Phi/\hbar}. \tag{56}$$

As a specific case of the equation (54), it is seen that $e^{-\Phi/\hbar}$ is an idempotent operator under the star product,

$$e^{-\Phi(z,\bar{z})/\hbar} * e^{-\Phi(z,\bar{z})/\hbar} = e^{-\Phi(z,\bar{z})/\hbar} e^{-\Phi(0,\bar{z})/\hbar} = e^{-\Phi(z,\bar{z})/\hbar}, \tag{57}$$

where $\Phi(0,\bar{z}) = 0$ is used.

By using the relations (54) and (56), it is possible to calculate explicitly star products containing $e^{-\Phi/\hbar}$ as follows,

$$\begin{aligned}
e^{-\Phi/\hbar} * \left(\partial_{i_1}\Phi(z,\bar{z})\cdots\partial_{i_n}\Phi(z,\bar{z})\right) &= e^{-\Phi/\hbar} \left(\partial_{i_1}\Phi(0,\bar{z})\cdots\partial_{i_n}\Phi(0,\bar{z})\right) \\
&= \bar{z}^{i_1}\cdots\bar{z}^{i_n} e^{-\Phi/\hbar} \\
&= e^{-\Phi/\hbar} * \bar{z}^{i_1} * \cdots * \bar{z}^{i_n},
\end{aligned} \tag{58}$$

$$\begin{aligned}
\left(\partial_{\bar{i}_1}\Phi(z,\bar{z})\cdots\partial_{\bar{i}_n}\Phi(z,\bar{z})\right) * e^{-\Phi/\hbar} &= z^{i_1}\cdots z^{i_n} e^{-\Phi/\hbar} \\
&= z^{i_1} * \cdots * z^{i_n} * e^{-\Phi/\hbar}.
\end{aligned} \tag{59}$$

We then consider a class of functions

$$M_{i_1\cdots i_m; j_1\cdots j_n} = \frac{z^{i_1}\ldots z^{i_m}\bar{z}^{j_1}\ldots\bar{z}^{j_n}}{\sqrt{m!n!\alpha_m(\hbar)\alpha_n(\hbar)}}\, e^{-\Phi/\hbar}, \tag{60}$$

where $\alpha_n(\hbar)$ is defined in (30). $M_{i_1\cdots i_m;j_1\cdots j_n}$ is totally symmetric under permutations of i's and j's, respectively. By using the commutation relations (49) and the fact that $e^{-\Phi/\hbar}$ is the vacuum projection, it can be shown that these functions form a closed algebra:

$$M_{i_1\cdots i_m;j_1\cdots j_n} * M_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr}\delta^{k_1\cdots k_n}_{j_1\cdots j_n} M_{i_1\cdots i_m;l_1\cdots l_s}, \tag{61}$$

where $\delta^{k_1\cdots k_n}_{j_1\cdots j_n}$ is defined as

$$\delta^{k_1\cdots k_n}_{j_1\cdots j_n} = \frac{1}{n!}\left[\delta^{k_1}_{j_1}\cdots\delta^{k_n}_{j_n} + \text{permutations of } (j_1,\cdots,j_n)\right]. \tag{62}$$

Therefore a set of linear combinations of $M_{i_1\cdots i_m;j_1\cdots j_n}$ forms a closed algebra. In particular, the operators $P_{i_1\cdots i_n} = M_{i_1\cdots i_n;i_1\cdots i_n}$ constitute a set of orthogonal projection operators,

$$P_{i_1\cdots i_m} * P_{j_1\cdots j_n} = \delta_{mn}\delta^{i_1\cdots i_n}_{j_1\cdots j_n} P_{i_1\cdots i_n}. \tag{63}$$

These operators would be useful for construction of solitons in field theories on the noncommutative $\mathbb{C}P^N$.

The star products between $M_{i_1\cdots i_m;j_1\cdots j_n}$ and one of $z^k, \partial_k\Phi, \bar{z}^k$ and $\partial_{\bar{k}}\Phi$ are explicitly calculated as follows,

$$z^k * M_{i_1\cdots i_m;j_1\cdots j_n} = \sqrt{\frac{m+1}{-m+1/\hbar}} M_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{64}$$

$$\partial_k\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{\frac{-m+1+1/\hbar}{m}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots \hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{65}$$

$$\bar{z}^k * M_{i_1\cdots i_m;j_1\cdots j_n} = \frac{1}{\sqrt{m(-m+1+1/\hbar)}} \sum_{l=1}^{m} \delta_{ki_l} M_{i_1\cdots \hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{66}$$

$$\partial_{\bar{k}}\Phi * M_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{(m+1)(-m+1/\hbar)} M_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{67}$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * z^k = \frac{1}{\sqrt{n(-n+1+1/\hbar)}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots \hat{j_l}\cdots j_n}, \tag{68}$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_k\Phi = \hbar\sqrt{(n+1)(-n+1/\hbar)} M_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{69}$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \bar{z}^k = \sqrt{\frac{n+1}{-n+1/\hbar}} M_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{70}$$

$$M_{i_1\cdots i_m;j_1\cdots j_n} * \partial_{\bar{k}}\Phi = \hbar\sqrt{\frac{-n+1+1/\hbar}{n}} \sum_{l=1}^{n} \delta_{kj_l} M_{i_1\cdots i_m;j_1\cdots \hat{j_l}\cdots j_n}. \tag{71}$$

5. Star product on $\mathbb{C}\mathbf{H}^{\mathbf{N}}$

In this section, we give explicit expressions for a star product on $\mathbb{C}H^N$, which is a noncompact Kähler manifold. As far as we know, star products in quantum deformation of $\mathbb{C}H^N$ $(n \geq 2)$ has not been explicitly constructed so far.

The Kähler potential of $\mathbb{C}H^N$ is given by

$$\Phi = -\ln\left(1 - |z|^2\right). \tag{72}$$

The metric $g_{i\bar{j}}$ and the inverse metric $g^{\bar{i}j}$ are defined by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1-|z|^2)\delta_{ij} + \bar{z}^i z^j}{(1-|z|^2)^2}, \tag{73}$$

$$g^{\bar{i}j} = (1-|z|^2)\left(\delta_{ij} - \bar{z}^i z^j\right). \tag{74}$$

The operator $L_{\bar{z}^l}$ is expanded as a power series of the noncommutative parameter $\hbar$,

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n B_n. \tag{75}$$

We assume that B_n has the following form,

$$B_n = \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}. \tag{76}$$

The factor $(-1)^{n-1}$ in the front of the coefficient $b_m^{(n)}$ is introduced for convenience.

Requiring $[L_{\bar{z}^l}, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0$, the following recursion relations which $b_m^{(n)}$ should satisfy are found

$$\begin{aligned} b_2^{(n)} &= b_2^{(n-1)} = \cdots = b_2^{(2)} = 1, \\ b_m^{(n)} &= b_{m-1}^{(n-1)} + (m-1) b_m^{(n-1)}. \end{aligned} \tag{77}$$

Hence $b_m^{(n)}$ coincides with $a_m^{(n)}$, and we obtain the explicit representation of the star product with separation of variables on $\mathbb{C}H^N$,

$$
\begin{aligned}
L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\
&= \bar{z}^l + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m(\hbar) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}, \qquad (78) \\
R_{z^l} &= z^l + \hbar D^{l} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{j_1}\Phi \cdots \partial_{j_{m-1}}\Phi D^{j_1} \cdots D^{j_{m-1}} D^{l} \\
&= z^l + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m(\hbar) \partial_{j_1}\Phi \cdots \partial_{j_{m-1}}\Phi D^{j_1} \cdots D^{j_{m-1}} D^{l}, \qquad (79)
\end{aligned}
$$

with

$$
\beta_n(t) = (-1)^n \alpha_n(-t) = \frac{\Gamma(1/t)}{\Gamma(n+1/t)}. \qquad (80)
$$

The differential operator corresponding to the left multiplication by a generic function f can be written in the forms of (38) and (40) with $c_n(\hbar) = \beta_n(\hbar)/n!$. And it can be shown that the Leibniz rule holds for the Killing vector fields corresponding to the $SU(N,1)$ isometry of $\mathbb{C}H^N$.

Using the representations of the star product, we can calculate the star products among z^i and $\bar{z}^i$,

$$
\begin{aligned}
z^i * z^j &= z^i z^j, \qquad (81) \\
z^i * \bar{z}^j &= z^i \bar{z}^j, \qquad (82) \\
\bar{z}^i * \bar{z}^j &= \bar{z}^i \bar{z}^j, \qquad (83) \\
\bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1-|z|^2) {}_2F_1\left(1,1;1+1/\hbar;|z|^2\right) \\
&\quad - \frac{\hbar}{1+\hbar} \bar{z}^i z^j (1-|z|^2) {}_2F_1\left(1,2;2+1/\hbar;|z|^2\right). \qquad (84)
\end{aligned}
$$

As in the case of $\mathbb{C}P^N$, $\{z^i, \partial_j\Phi\}$ and $\{\bar{z}^i, \partial_{\bar{j}}\Phi\}$ satisfy the commutation relations for the creation-annihilation operators. Also $e^{-\Phi/\hbar}$ is the vacuum projection operator,

$$
\partial_i\Phi * e^{-\Phi/\hbar} = 0, \qquad \bar{z}^i * e^{-\Phi/\hbar} = 0, \qquad (85)
$$

$$
e^{-\Phi/\hbar} * \partial_{\bar{i}}\Phi = 0, \qquad e^{-\Phi/\hbar} * z^i = 0, \qquad (86)
$$

and

$$
e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar}. \qquad (87)
$$

As in the case of $\mathbb{C}P^N$, we consider a class of functions

$$N_{i_1\cdots i_m;j_1\cdots j_n} = \frac{z^{i_1}\ldots z^{i_m}\bar{z}^{j_1}\ldots\bar{z}^{j_n}}{\sqrt{m!n!\beta_m(\hbar)\beta_n(\hbar)}}e^{-\Phi/\hbar} \tag{88}$$

$N_{i_1\cdots i_m;j_1\cdots j_n}$ is totally symmetric under permutations of i's and j's, respectively. Then we can show that these functions form a closed algebra

$$N_{i_1\cdots i_m;j_1\cdots j_n} * N_{k_1\cdots k_r;l_1\cdots l_s} = \delta_{nr}\delta^{k_1\cdots k_n}_{j_1\cdots j_n}N_{i_1\cdots i_m;l_1\cdots l_s}. \tag{89}$$

It is also shown easily that the operators $N_{i_1\cdots i_n;i_1\cdots i_n}$ are orthogonal projection operators.

Moreover, the star products between $N_{i_1\cdots i_m;j_1\cdots j_n}$ and one of $z^k, \partial_k\Phi, \bar{z}^k$ and $\partial_{\bar{k}}\Phi$ are calculated as follows,

$$z^k * N_{i_1\cdots i_m;j_1\cdots j_n} = \sqrt{\frac{m+1}{m+1/\hbar}}N_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{90}$$

$$\partial_k\Phi * N_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{\frac{m-1+1/\hbar}{m}}\sum_{l=1}^{m}\delta_{ki_l}N_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{91}$$

$$\bar{z}^k * N_{i_1\cdots i_m;j_1\cdots j_n} = \frac{1}{\sqrt{m(m-1+1/\hbar)}}\sum_{l=1}^{m}\delta_{ki_l}N_{i_1\cdots\hat{i_l}\cdots i_m;j_1\cdots j_n}, \tag{92}$$

$$\partial_{\bar{k}}\Phi * N_{i_1\cdots i_m;j_1\cdots j_n} = \hbar\sqrt{(m+1)(m+1/\hbar)}N_{ki_1\cdots i_m;j_1\cdots j_n}, \tag{93}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * z^k = \frac{1}{\sqrt{n(n-1+1/\hbar)}}\sum_{l=1}^{n}\delta_{kj_l}N_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n}, \tag{94}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \partial_k\Phi = \hbar\sqrt{(n+1)(n+1/\hbar)}N_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{95}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \bar{z}^k = \sqrt{\frac{n+1}{n+1/\hbar}}N_{i_1\cdots i_m;j_1\cdots j_n k}, \tag{96}$$

$$N_{i_1\cdots i_m;j_1\cdots j_n} * \partial_{\bar{k}}\Phi = \hbar\sqrt{\frac{n-1+1/\hbar}{n}}\sum_{l=1}^{n}\delta_{kj_l}N_{i_1\cdots i_m;j_1\cdots\hat{j_l}\cdots j_n}. \tag{97}$$

6. Summary and discussion

In this article, we provided explicit expressions of star products in $\mathbb{C}P^N$ and $\mathbb{C}H^N$ by using the method for making the deformation quantization with separation of variables which is proposed by Karabegov. In this method, a star product by functions is described by a formal series of differential operators, which is determined as solutions of an infinite system of differential equations. We constructed the explicit forms of solutions of the equations in the case of $\mathbb{C}P^N$ and $\mathbb{C}H^N$. Operators corresponding to the left (right) star multiplications of functions are obtained as power series of a noncommutative parameter $\hbar$ in which each term contains the Stirling numbers of the second kind, the Kähler potentials of the manifolds, and differential operators. The expressions of the star products obtained here depend on the metric of the manifolds, and it is seen that the Leibniz rule holds for the Killing vector fields corresponding to isometries of these manifolds.

We also constructed the Fock representations of the star products by using the fact that $\{z^i, \partial_j \Phi\}$ and $\{z^{\bar{i}}, \partial_{\bar{j}} \Phi\}$ constitute $2N$ sets of the creation-annihilation operators under the star product. We found the function $e^{-\Phi/\hbar}$ corresponding to the vacuum projection operator. Then we considered functions which are derived by multiplying polynomials of z^i and $z^{\bar{i}}$ on $e^{-\Phi/\hbar}$, and showed that these functions satisfy the closed algebra under the star product. In particular, we obtained the functions which constitute a set of orthogonal projection operators.

Can we apply our analysis performed here to other Kähler manifolds? For example, let us try to extend the covariant expression of L_f (40) to locally symmetric Kähler manifolds, $\nabla_\mu R_{\nu\rho\sigma}{}^\lambda = 0$. We assume the following form of L_f,

$$L_f g = \sum_{n=0}^{\infty} T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g \right), \tag{98}$$

where g is a scalar function. We assume that $T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n}$ is a covariantly constant tensor, $\nabla T_n = 0$, and completely symmetric under permutations of $\bar{j}$'s and k's, respectively. From the condition $[L_f, \partial_{\bar{i}} \Phi + \hbar \partial_{\bar{i}}] = 0$, the following recursion relations for $T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n}$ are found,

$$\begin{aligned} &\Big[n T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n} g_{k_n \bar{i}} - \hbar T_{n-1}^{\bar{j}_1 \cdots \bar{j}_{n-1}, k_1 \cdots k_{n-1}} \delta_{\bar{i}}^{\bar{j}_n} \\ &- \hbar \frac{n(n-1)}{2} T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_{n-2} pq} R_{\bar{i}pq}{}^{k_{n-1}} \Big] \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \right) \left(\nabla_{k_1} \cdots \nabla_{k_{n-1}} g \right) = 0. \end{aligned} \tag{99}$$

Since the recursion relations include only the metric and the Riemann tensor, $T_n^{\bar{j}_1 \cdots \bar{j}_n, k_1 \cdots k_n}$ is determined as a function of these quantities and the assumption $\nabla T_n = 0$ is satisfied. Though it is difficult to derive solutions of the recursion relations in closed form expressions, solutions can be determined order by order. In the case of $\mathbb{C}P^N$, the recursion relations are simplified. Because $R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}} g_{k\bar{l}} - g_{i\bar{l}} g_{k\bar{j}}$ on $\mathbb{C}P^N$ with the metric (17), it can be shown that L_f has the covariant form (40).

It is interesting to study field theories on the noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$ constructed in this article. For example, we can define a scalar field on $\mathbb{C}P^N$ as

$$\phi = \sum \phi_{i_1 \cdots i_m; j_1 \cdots j_n} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

where $M_{i_1 \cdots i_m; j_1 \cdots j_n}$ is defined in (60). Since the algebra which M's satisfy is known, we can calculate quantities constructed by ϕ. As an action, we can choose, e.g.

$$\int d\mu \left(\frac{1}{2} \mathcal{L}_a \phi \mathcal{L}_a \phi + V(\phi) \right)$$

where $\mathcal{L}_a$ is the Killing vector field (46) and $d\mu$ is the usual integration measure with respect to the Fubini-Study metric on $\mathbb{C}P^N$. It is important to construct noncommutative solitons as classical solutions and to analyse quantum properties of these theories.

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