Aspects of Quantum Modes on de Sitter Spacetime

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Abstract

General aspects of the quantum modes functions of the scalar fields on de Sitter spacetime are reviewed. Spherical eigenfunctions of a Hamiltonian operator, found using a special technique that involves time evolution pictures are discussed on the spatially flat FLRW chart. A concise background on de Sitter spacetime is provided, together with an argument on what coordinate charts are the most suitable for the endeavor of determining the quantum modes, as a first step in constructing a quantum field theory on this curved background.

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1. Introduction

The de Sitter spacetime is a positively curved maximal symmetric manifold. It has the same high symmetry as Minkowski spacetime, and it describes an expanding spacetime, empty of matter or radiation, but having a positive cosmological constant. For these features, it is promising as a classical background on which to formulate a quantum field theory, provided certain difficulties are overcome. Alternatively, it is used in cosmology to model the expansion of the inflationary phase of the Universe.
Thanks to its high symmetry, the de Sitter manifold can be covered with a multitude of coordinate charts. [1, 2, 3, 4, 5]. On some charts, the field equations have separating variables, and consequently mode functions can be expressed. Scalar waves have been discussed on de Sitter in various coordinates, as early as 1938 by Schrödinger [6, 7] as classical waves, and then quantum waves throughout the years [8, 9, 10, 11, 12].
Since the de Sitter spacetime is as symmetrical as Minkowski spacetime, it has also been interpreted as an extension of special relativity to a spacetime empty of matter, but having a positive cosmological constant. [13, 14, 15] Therefore, the study of the various solutions of the field equations is important, since it is the first step needed in order to extend the Minkowskian

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quantum field theory to a curved spacetime like de Sitter. However, the opinions on how to proceed in building an interacting quantum field theory are diverse, and sometimes conflicting. [16, 17, 18, 19]
In this paper, we first introduce de Sitter spacetime as a manifold, covered with certain coordinate charts, arguing for the advantages of the most popular ones. Then, general considerations on the determination of the scalar quantum modes are discussed. In the end, some quantum modes that are eigenfunctions of a Hamiltonian-like operator are presented, showing that more than one set of mode can be introduced in a chart. The procedure by which these result can be obtained is also discussed.

2. The de Sitter manifold

The de Sitter manifold is a Lorentzian manifold, taken here to be of dimension $3+1$. It is usually defined as being embedded in a $4+1$ dimensional flat spacetime (termed "global embedding Minkowski spacetime"- GEMS, or simply "embedding space"), satisfying a constraint of the type

$$\eta_{AB}Z^A Z^B = -\frac{1}{\omega^2}\,, \tag{1}$$

where Z_A with $A = 1..5$ are the coordinates in the embedding space, and $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1,-1)$ the flat Lorentzian metric.
We assume that the de Sitter manifold is covered with a local chart $\{x^\mu\}$, $\mu = 1..4$, such that

$$Z_A = Z_A(x^\mu), \tag{2}$$

are called the embedding functions, and contain all the information about the embedding.
Then, the metric tensor on dS is inherited through the embedding $Z^A(x^\mu)$ and reads

$$g_{\mu\nu} = \eta_{AB}\frac{\partial Z^A}{\partial x^\mu}\frac{\partial Z^B}{\partial x^\nu}\,. \tag{3}$$

The de Sitter manifold, just as the Minkowski manifold is maximally symmetric (it has the maximum amount of Killing vectors- 10 for the total 4 dimensions). The Killing vectors components on de Sitter can be expressed in terms of the embedding, being inherited also from the embedding space:

$$k^\mu_{AB} = g^{\mu\nu}\eta_{AC}\eta_{BD}\left(Z^C\frac{\partial Z^D}{\partial x^\nu} - Z^D\frac{\partial Z^C}{\partial x^\nu}\right). \tag{4}$$

Now, we can turn to the different coordinates used to cover this manifold. One of the most used charts in this respect are the Friedmann-Lemaitre-Robertson-Walker charts. De Sitter is the only manifold that admits coordinatizations of all the three FLRW types. These are important in cosmology, because they exhibit isotropy and homogeneity.
Of these three, the spatially flat (Euclidean) chart $\{t,x,y,z\}$ presents the most interest because of observations that indicate that even though there

is spacetime curvature, the spatial part is flat- it's time slices are Euclidean 3-spaces. Moreover, it exhibits manifest translation symmetry:

$$ds^2 = dt^2 - e^{-2\omega t}(dx^2 + dy^2 + dz^2). \tag{5}$$

Another important chart is the de Sitter static chart $\{t_s, r_s, \theta, \phi\}$which has the ∂_t is a Killing vector, since none of the components of its metric tensor are time-dependent. It would be useful if a chart can be found that has the best of both worlds: ∂_t as a Killing vector and it's time slices being Euclidean spaces. This can be thought as a hybrid between the spatially flat FLRW and the static chart. It is sometimes termed "de Sitter-Painlevé" chart, for its analogy with the Painlevé chart on the Schwarzschild manifold. [20] Its line element is

$$ds^2 = (1 - \omega^2 r_s^2)dt^2 + 2\omega r_s dr_s dt - dr_s^2 - r_s^2 d\Omega_2^2. \tag{6}$$

All the above charts have a useful property: they can separate the Klein-Gordon equation. There are of course many more charts that don't have the property, but nevertheless receive attention. A worthy mention would be a class of charts that admit parameterizations of the form

$$Z^\mu(x^\mu) = \frac{x^\mu}{f(x^\nu x_\nu)}\,, \tag{7}$$

because they put time and space on an equal footing as coordinates. These have been vehiculated as good candidates in the formulation of an extension of special relativity from Minkowski spacetime to de Sitter spacetime. For instance, the Beltrami chart [21], where $f(x) = \sqrt{1 - \omega^2(t^2 - \vec{x}^2)}$ gives rise to a coordinate chart that are interpreted as the "inertial coordinates" of de Sitter spacetime. Also, the stereographic chart [14] with $f(x) = 1 - \omega^2(t^2 - \vec{x}^2)/4$, a chart in which the metric tensor is conformal to that of Minkowski spacetime, and used as the basis for "de Sitter special relativity", an extension for non-vanishing cosmological constant to the traditional special relativity. [15]

However, one serious drawback is that no symmetries are manifest, and consequently the field equations do not separate in either of these charts, though the Laplace equation for the spatial coordinates does. [22] However, a solution, including an approximation was proposed for the Beltrami chart. [23]

3. The quantum modes

Having established the background spacetime as the de Sitter spacetime, it is evident that gravity remains classical. The background is fixed, and as such the gravitational field is classical (not quantized). The de Sitter background is viewed as the arena where interactions in a quantum theory arise. Therefore, its main attribute is to shape the wave functions of the fields on it.

Passing from a Minkowski background to a de Sitter one, a few things are to be noticed: first, the numbers of Killing vectors stays the same (10 for a 3+1 dimensional manifold), so that the setting is still a maximal dimensional spacetime. The introduction of the parameter ω that denotes the cosmological (Hubble's) constant also implies the quantities must reduce to the Minkowski ones when this parameter vanishes. But its introduction increases the complexity of the field equations, and the first step in a quantum theory is to find the solutions to these equations.
First, the 10 symmetries are represented by the 10 Killing vectors having the components given by relation (4). Each of these corresponds in the quantum theory to a conserved operator, which in the scalar field case is

$$X_{AB} = -ik^{\mu}_{AB}\partial_{\mu}. \tag{8}$$

Specifically, the operators are:a Hamiltonian $H = \omega X_{04}$, three components of the angular momentum operator $J_i = i\epsilon_{ijk}X_{jk}$, momentum $P_i = \omega(X_{i4} + X_{0i})$ and its dual $Q_i = \omega(X_{i4} - X_{0i})$. Also, noteworthy are the components of a transvection operator $R_i = X_{i4}$, also called 'Runge-Lenz-type' operator. [24] Two comments are here in order- first, why is H called a Hamiltonian, if on de Sitter there is no global timelike Killing vector, and this casted doubts on the concept of energy defined in a de Sitter spacetime [25]- it is because in some charts, for instance the spatially flat FLRW chart, ∂_t is timelike, everywhere the observer in the origin can make measurements. And secondly, why the choice for the momentum operator is $\mathbf{P}$ and not $\mathbf{R}$. In the flat limit, both give the expected Minkowski result:

$$\lim_{\omega\to 0} P_i = \lim_{\omega\to 0} \omega R_i = P_i^{\mathbb{M}} \equiv -i\partial_i\,. \tag{9}$$

However, by looking at the commutation relations $[P_i, P_j] = 0$, we can observe that $\mathbf{P}$ is an abelian operator, and respectively $[R_i, R_j] = i\epsilon_{ijk}J_k$- from where it can be seen why it was said about $\mathbf{R}$ that it is a Runge-Lenz type operator- its components form an $o(4)$ algebra along with the components of the angular momentum operator $\mathbf{J}$.
It can be deduced that the commutation relation between the Hamiltonian and the momentum operator components is $[H, P_i] = i\omega P_i$. This means that energy and momentum cannot be measured simultaneously on de Sitter spacetime. There is a Heisenberg-like uncertainty relation between these two observables. As such, H and $\mathbf{P}$ will not share common eigenfunctions. The scalar field operator can be expanded as

$$\Phi(x) = \int da\, db\, dc \quad f_{abc}(x)a(a,b,c) + f^{*}_{abc}(x)a^{\dagger}(a,b,c), \tag{10}$$

and it must obey a field equation $\mathcal{E}\Phi(x) = 0$. In the scalar case, this is the Klein-Gordon equation

$$\frac{1}{\sqrt{|g|}}\,\partial_{\mu}(\sqrt{|g|}g^{\mu\nu}\partial_{\nu}\Phi(x)) - m^2\Phi(x) = 0. \tag{11}$$

It must also obey a number of eigenvalue equations for a selected number of operators A, B, C, which together with the field equation operator $\mathcal{E}$ form a complete set of commuting operators (CSCO) $\{\mathcal{E}, A, B, C\}$:

$$A\Phi(x) = a\Phi(x), \qquad B\Phi(x) = b\Phi(x), \qquad C\Phi(x) = c\Phi(x). \tag{12}$$

The eigenvalues of the CSCO behave as separating constants for the solutions, and their physical interpretation is that they are the values arising from a measurement performed in accordance to the operators of the CSCO. For the de Sitter spacetime covered with the Euclidean chart, this relates to solving the equation

$$\left(\partial_t^2 + 3\omega\partial_t - e^{-2\omega t}\Delta_{x,y,z} + m^2\right)\Phi(t,\vec{x}) = 0, \tag{13}$$

In this chart, the momentum operator components have the simple expression

$$P_i = -i\partial_i, \tag{14}$$

while the Hamiltonian is

$$H = i\partial_t - i\omega x^i\partial_i. \tag{15}$$

The straightforward solutions in this case are plane waves. They were found by Nachtmann [8], and these are the ones corresponding to the CSCO $\{\mathcal{E}, \mathbf{P}\}$. However, there is another possibility for plane waves, as shown in Ref. [11], corresponding to a CSCO that includes the H operator, in which the authors used a time evolution technique in order to bring the form of the Hamiltonian operator to ∂_t, called "the Schrödinger Picture" (SP). [26] This technique is the consequence of the existence of the coordinate chart (6).
In this picture, both the forms of the wave functions, and the operators change as

$$\Phi(x) \to \Phi_S(x) = U(x)\Phi(x), \tag{16}$$

$$O \to O_S = U(x)OU(x)^{-1}, \tag{17}$$

where

$$U(x) = e^{-\omega t(x^i\partial_i)}. \tag{18}$$

Particularly, expressions transform like

$$U(x)F(x^i)U(x)^{-1} = F(e^{-\omega t}x^i), \tag{19}$$

$$U(x)F(\partial_i)U(x)^{-1} = F(e^{\omega t}\partial_i), \tag{20}$$

such that a coordinate transformation from the Euclidean coordinates to the Painlevé ones is said to be equivalent to a change of picture in this formalism.

4. Energy basis spherical modes

In the previous section, a set of new plane waves that are eigenfunctions of the Hamiltonian were mentioned, together with the procedure used to determine their mathematical forms. [11] Formally, they correspond to a CSCO $\{\mathcal{E}, H, \mathcal{N}_1, \mathcal{N}_2\}$, where the operators $\mathcal{N}_i$ are formal non-differential operators that encode the direction of the momentum vector. Similarly, one can find the spherical waves by solving the Klein-Gordon equation, applying the SP formalism, but this time with a CSCO $\{\mathcal{E}, H, \mathbf{L}^2, L_z\}$ genuinely made up only of differential operators. [27]
First, let us formulate the Klein-Gordon equation in SP, the following concrete transformations, which result literally from (19-20):

$$\begin{aligned} \partial_t &\rightarrow \partial_t + \omega x^i \partial_i, && (21)\\ \partial_i &\rightarrow e^{\omega t}\partial_i, && (22)\\ \Delta &\rightarrow e^{2\omega t}\Delta, && (23)\\ \Phi(t,\vec{x}) &\rightarrow \Phi_S(t,\vec{x}), && (24) \end{aligned}$$

which lead to the Klein-Gordon equation in the same chart, but in SP. Then, by passing from the Cartesian coordinates $\{t, \vec{x}\}$ to spherical coordinates $\{t, r, \theta, \phi\}$, the following equation is obtained:

$$\left((\partial_t + \omega r\partial_r)^2 + 3\omega(\partial_t + \omega r\partial_r) - \partial_r^2 - \frac{2}{r}\partial_r - \frac{\Delta_{\theta,\phi}}{r^2} + m^2\right)\Phi_S(t,r,\theta,\phi) = 0. \tag{25}$$

The energy basis spherical quantum modes are solutions of this equation, and also eigenfunctions of the following operators: $H, \vec{L}^2, L_z$, that insure the separation of variables. Solving the eigenvalues equations for these equations in the SP, for the following CSCO $H = i\partial_t, \vec{L}^2 = -\Delta_{\theta,\phi}, L_z = -\Delta_\phi$, it can be shown that the solution of the equation is separable as:

$$f^S_{E,l,m_l}(t,r,\theta,\phi) = Ne^{-iEt}R^S_{E,l}(r)Y_{l,m_l}(\theta,\phi), \tag{26}$$

where $Y_{l,m_l}(\theta,\phi)$ are the two dimensional spherical harmonics.
The separation of variables occurs only in the SP, and this is the result of applying the aforementioned time evolution picture technique. [11] In the SP, $H = i\partial_t$ is conserved, while in the NP the modified Hamiltonian $H = i\partial_t - i\omega x^i\partial_i$ is conserved. The radial part of the Klein-Gordon equation in SP is given by an ordinary differential equation

$$\begin{aligned} \Bigg[(\omega^2r^2 - 1)\frac{d^2}{dr^2} &+ \left(4\omega^2 r - 2i\omega E r - \frac{2}{r}\right)\frac{d}{dr} \\ &-E^2 - 3i\omega E + \frac{l(l+1)}{r^2} + m^2\Bigg] R^S_{E,l}(r) = 0, \end{aligned} \tag{27}$$

which by discarding the solution that is singular at the origin and passing from SP to NP, has the solution

$$
\begin{aligned}
f_{E,l,m_l}(t,r,\theta,\phi) = & \, N e^{-iEt}(\omega r e^{\omega t})^l Y_{l,m_l}(\theta,\phi) \\
& \times {}_2F_1\left(\sigma_+ + \frac{l}{2}, \sigma_- + \frac{l}{2}; l + \frac{3}{2}; \omega^2 r^2 e^{2\omega t}\right),
\end{aligned} \tag{28}
$$

where $\sigma_\pm = \frac{3}{4} - \frac{i\epsilon}{2} \pm \frac{\nu}{2}$ and $\nu = i\sqrt{\mu^2 - \frac{9}{4}}$ and N is a normalization constant, which can be found by writing the solution as a Hankel transform of another function, which allows writing quantum modes in the integral representation [27]

$$
\begin{aligned}
f_{E,l,m_l}(t,r,\theta,\phi) = & \, N 2^{i\epsilon} i^{-\frac{1}{2}-l} \omega^{\frac{3}{2}} e^{-\frac{\epsilon\pi}{2}} \pi e^{\frac{i\pi\nu}{2}} \frac{\Gamma\left(l+\frac{3}{2}\right)}{\Gamma\left(\sigma_+ + \frac{l}{2}\right)\Gamma\left(\sigma_- + \frac{l}{2}\right)} \\
& \times e^{-\frac{3\omega t}{2}} Y_{l,m_l}(\theta,\phi) \frac{1}{\sqrt{r}} \int_0^\infty s^{-i\epsilon} H_\nu^{(1)}(s e^{-\omega t}) J_{l+\frac{1}{2}}(\omega r s) ds,
\end{aligned} \tag{29}
$$

which, by using scalar product gives the desired normalization constant

$$
N = \frac{e^{\frac{\epsilon\pi}{2}}}{2\sqrt{2}\pi\omega^{\frac{3}{2}}} \frac{\Gamma\left(\sigma_+ + \frac{l}{2}\right)\Gamma\left(\sigma_- + \frac{l}{2}\right)}{\Gamma\left(l+\frac{3}{2}\right)}. \tag{30}
$$

Summing up, the definitive expression of the spherical energy basis quantum modes in the hypergeometric form is

$$
\begin{aligned}
f_{E,l,m_l}(t,r,\theta,\phi) = & \frac{1}{2\sqrt{2}\pi\omega^{\frac{3}{2}}} \frac{\Gamma\left(\sigma_+ + \frac{l}{2}\right)\Gamma\left(\sigma_- + \frac{l}{2}\right)}{\Gamma\left(l+\frac{3}{2}\right)} e^{-iEt}(\omega r e^{\omega t})^l \\
& \times {}_2F_1\left(\sigma_+ + \frac{l}{2}, \sigma_- + \frac{l}{2}; l + \frac{3}{2}; \omega^2 r^2 e^{2\omega t}\right) Y_{l,m_l}(\theta,\phi).
\end{aligned} \tag{31}
$$

5. Conclusion

In this paper, we have discussed a few parameterizations on the de Sitter hyperboloid and the relevance of the higher dimensional embedding and symmetry in the determination of the quantum modes. A conclusion is that the most useful charts for computing quantum modes are the ones where symmetries are manifest. For example, on the spatially flat FLRW chart, translational symmetries are manifest- plane waves can be defined that could be used to write a de Sitter interacting quantum field theory with a perturbative Feynman-Dyson formalism. [28, 29]

It has also been argued that on a chart there can be more than one useful mode-expansion of the field operator, though it is not straightforward to express them. The example of finding spherical energy eigenfunctions on the spatially flat de Sitter chart [27] is reviewed, using the time evolution picture technique known as the "Schrödinger Picture" formalism. These spherical modes could potentially be used for partial wave analysis on de Sitter spacetime. Of note is that while the energy-basis modes are different from the momentum-basis ones, there is still no Bogolyubov mixing and therefore the vacuum is stable under transformations from one set to another.

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