Algebraic Structures in Renormalization

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Abstract

We consider the problem of preservation of symmetries in the perturbative renormalization from the perspective of the deformation theory.

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We start with a general introduction to the concept of renormalization. Let us have a "physical quantity",

$$U(\kappa_1, \dots, \kappa_N; \varepsilon) \equiv U(\underline{\kappa}; \varepsilon) \qquad (\underline{\kappa} = (\kappa_1, \dots, \kappa_N)) \tag{1}$$

that depends on a set of physical parameters $\kappa_1, \dots, \kappa_N$, called also *coupling constants*, and one more subsidiary parameter called a *regularization parameter*. The meaning of the coupling constants is that they introduce a deformation of some initial physical model at $\kappa_1 = \dots = \kappa_N = 0$, which for instance, can present the physical system "without interaction". The regularization parameter is introduced in order to make the quantity U mathematically well defined. Let us set $\varepsilon \to 0$ to be the limit that should correspond to the actual physical model, but on the other hand, let us assume that

$$\nexists \lim_{\varepsilon \to 0} U(\underline{\kappa}; \varepsilon) \,.$$

We note that the values of $U(\underline{\kappa}; \varepsilon)$ can be just numbers but more often they belong to some vector space $\mathfrak{V}$. For instance, in Quantum Field Theory (QFT) these $U(\underline{\kappa}; \varepsilon)$ can be the cross sections or the correlation functions: in these cases we consider the value of $U(\underline{\kappa}; \varepsilon)$ to be the corresponding cross section or correlation function, respectively, as an element of some space of functions (or distributions). In the latter case we dismiss the extra dependencies in our notation $U(\underline{\kappa}; \varepsilon)$, hiding them in the vector space $\mathfrak{V}$ of the values of U.

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The general idea of the renormalization is to replace the coupling constants $\underline{\kappa}$ in (1) with some "unphysical" parameters $\underline{\kappa}'$ that are called unrenormalized (or, "bare") coupling constants and set them to be some (still unknown) functions in the actual physical parameters $\underline{\kappa}$,

$$\underline{\kappa}' := \underline{\mathcal{K}}(\underline{\kappa};\varepsilon) \quad (\underline{\mathcal{K}}(\underline{\kappa};\varepsilon) = (\mathcal{K}_i(\underline{\kappa};\varepsilon))_{i=1}^N)$$

depending on ε. Then the renormalization problem is to achieve a finite limit:

$$\exists \lim_{\varepsilon\to 0} U(\underline{\mathcal{K}}(\underline{\kappa};\varepsilon);\varepsilon) =: \lim_{\varepsilon\to 0} U^{\mathrm{ren}}(\underline{\kappa};\varepsilon) =: U^{\mathrm{ren}}(\underline{\kappa}) . \tag{2}$$

This limit, namely, is the renormalized physical quantity $U^{\mathrm{ren}}(\underline{\kappa})$. There is somewhat more general concept of renormalization that affects also the values of $U^{\mathrm{ren}}(\underline{\kappa};\varepsilon)$:

$$\exists \lim_{\varepsilon\to 0} \mathcal{Z}(\underline{\kappa};\varepsilon)\cdot U(\underline{\mathcal{K}}(\underline{\kappa};\varepsilon);\varepsilon) =: \lim_{\varepsilon\to 0} U^{\mathrm{ren}}(\underline{\kappa};\varepsilon) =: U^{\mathrm{ren}}(\underline{\kappa}) , \tag{3}$$

where $\mathcal{Z}(\underline{\kappa};\varepsilon)$ is an automorphism of $\mathfrak{V}$ (and "$\cdot$" stands for its action on $\mathfrak{V}$). Most simply, $\mathcal{Z}(\underline{\kappa};\varepsilon)$ can be just a numerical factor that reflects the renormalization of the "scale of units" in which U is measured. However, we shall start with the case (2), i.e., without the additional multiplicative renormalization in (3).

We can proceed further in solving the so stated renormalization problem within the framework of "perturbation theory". In this case we consider $U(\underline{\kappa}';\varepsilon)$ as a formal power series in $\underline{\kappa}'$:

$$U(\underline{\kappa}';\varepsilon) = U^{(0)}(\varepsilon) + \sum_{i=1}^N U_i^{(1)}(\varepsilon)\,\kappa_i' + \sum_{i_1,i_2=1}^N U_{i_1,i_2}^{(2)}(\varepsilon)\,\kappa_{i_1}'\kappa_{i_2}' + \cdots .$$

We will try to solve the renormalization problem order by order. We assume that the zeroth and the first orders are free of divergences and are nonzero:

$$\exists \lim_{\varepsilon\to 0} U^{(0)}(\varepsilon) \neq 0 , \qquad \exists \lim_{\varepsilon\to 0} U_i^{(1)}(\varepsilon) \neq 0 . \tag{4}$$

If $\nexists \lim\limits_{\varepsilon\to 0} U_{i_1,i_2}^{(2)}(\varepsilon)$ we can try with a substitution $U(\underline{\mathcal{K}}(\underline{\kappa};\varepsilon);\varepsilon)$ with

$$\underline{\mathcal{K}}(\underline{\kappa};\varepsilon) = (\mathcal{K}_i(\underline{\kappa};\varepsilon))_{i=1}^N , \quad \mathcal{K}_i(\underline{\kappa};\varepsilon) = \kappa_i + \sum_{i_1,i_2=1}^N \mathcal{K}_{i;i_1,i_2}^{(2)}(\varepsilon)\,\kappa_{i_1}\kappa_{i_2} + \cdots$$

and obtain

$$\begin{aligned} U(\underline{\mathcal{K}}(\underline{\kappa};\varepsilon);\varepsilon) &= U^{(0)}(\varepsilon) + \sum_{i=1}^N U_i^{(1)}(\varepsilon)\,\kappa_i \\ &+ \sum_{i_1,i_2=1}^N \Big(U_{i_1,i_2}^{(2)}(\varepsilon) + \sum_{i=1}^N U_i^{(1)}(\varepsilon)\,\mathcal{K}_{i;i_1,i_2}^{(2)}(\varepsilon) \Big)\,\kappa_{i_1}\kappa_{i_2} + \cdots , \end{aligned}$$

i.e., the zeroth and the first orders are not affected and the second order takes an additive correction. We can try then to choose the latter additive correction so that it cancels the divergence in $U^{(2)}_{i_1,i_2}(\varepsilon)$. In other words, *$U(\underline{\kappa};\varepsilon)$ is perturbatively renormalizable at the second order* iff *the "divergent part" of $U^{(2)}_{i_1,i_2}(\varepsilon)$ is representable as a linear combination of first order terms $U^{(1)}_i(\varepsilon)$* (with ε–dependent coefficients).

In case (3) we set in addition

$$\mathcal{Z}(\underline{\kappa};\varepsilon) \,=\, 1 + \sum_{i_1,i_2\,=\,1}^{N} \mathcal{Z}^{(2)}_{i_1,i_2}(\varepsilon)\, \kappa_{i_1}\kappa_{i_2} + \cdots$$

and we get

$$\mathcal{Z}(\underline{\kappa};\varepsilon)\cdot U(\underline{\mathcal{K}}(\underline{\kappa};\varepsilon);\varepsilon) \,=\, U^{(0)}(\varepsilon) + \sum_{i\,=\,1}^{N} U^{(1)}_i(\varepsilon)\,\kappa_i + \sum_{i_1,i_2\,=\,1}^{N} \Big(U^{(2)}_{i_1,i_2}(\varepsilon) + \sum_{i\,=\,1}^{N} U^{(1)}_i(\varepsilon)\,\mathcal{K}^{(2)}_{i;i_1,i_2}(\varepsilon) + \mathcal{Z}^{(2)}_{i_1,i_2}(\varepsilon)\cdot U^{(0)}(\varepsilon)\Big)\,\kappa_{i_1}\kappa_{i_2} + \cdots\,.$$

We see now that we get one more additive correction to the second order, $\mathcal{Z}^{(2)}_{i_1,i_2}(\varepsilon)\cdot U^{(0)}(\varepsilon)$, which however can trivialize it (i.e., can cancel it completely) if we allow $\mathcal{Z}^{(2)}_{i_1,i_2}(\varepsilon)$ to be an arbitrary automorphism of the vector space $\mathfrak{V}$ of values of $U(\underline{\kappa};\varepsilon)$. In the case when $\mathcal{Z}(\underline{\kappa};\varepsilon)$ is a series with numerical coefficients we get the conclusion that *$U(\underline{\kappa};\varepsilon)$ is perturbatively renormalizable at the second order* iff *the "divergent part" of $U^{(2)}_{i_1,i_2}(\varepsilon)$ is representable as a linear combination of first and zeroth order terms of $U(\underline{\kappa};\varepsilon)$.*

Then, by induction we assume that we have constructed a sequence of formal power series $U = U_1,\, U_2,\, \dots,\, U_M,\, \dots$:

$$U_M(\underline{\kappa}';\varepsilon) \,=\, \sum_{n\,=\,0}^{\infty}\ \sum_{i_1,\dots,i_n\,=\,1}^{N} U^{(n)}_{M\,|\,i_1,\dots,i_n}(\varepsilon)\,\kappa'_{i_1}\,\cdots\,\kappa'_{i_n}$$

so that

$$\exists \lim_{\varepsilon\to 0} U^{(n)}_{M;i_1,\dots,i_n}(\varepsilon) \qquad \text{for} \qquad n \,=\, 0,1,\dots,M\,.$$

To subtract the divergence at order $M+1$ of $U_M(\underline{\kappa};\varepsilon)$ we perform a substitution and multiplication by

$$\kappa'_i = \mathcal{K}_{M\,|\,i}(\underline{\kappa};\varepsilon) = \kappa_i + \sum_{i_1,\dots,i_{M+1}\,=\,1}^{N} \mathcal{K}^{(M+1)}_{M\,|\,i;i_1,\dots,i_{M+1}}(\varepsilon)\,\kappa_{i_1}\cdots\kappa_{i_{M+1}} + \cdots\,,$$

$$\mathcal{Z}_M(\underline{\kappa};\varepsilon) = 1 + \sum_{i_1,i_2=1}^{N} \mathcal{Z}^{(M+1)}_{M\,|\,i;i_1,\dots,i_{M+1}}(\varepsilon)\,\kappa_{i_1}\cdots\kappa_{i_{M+1}} + \cdots\,,$$

respectively. Then again in the result

$$U_{M+1}(\underline{\kappa};\varepsilon) \;:=\; \mathcal{Z}_M(\underline{\kappa};\varepsilon)\cdot U_M(\underline{\mathcal{K}}_M(\underline{\kappa};\varepsilon);\varepsilon)$$

the orders below $M+1$ are not affected and the $(M+1)$st order gets an additive correction:

$$U^{(M+1)}_{M+1\,|\,i_1,\dots,i_{M+1}}(\varepsilon) \;=\; U^{(M+1)}_{M\,|\,i_1,\dots,i_{M+1}}(\varepsilon) + \sum_{i=1}^{N} U^{(1)}_i(\varepsilon)\,\mathcal{K}^{(M+1)}_{M\,|\,i;i_1,\dots,i_{M+1}}(\varepsilon) + \mathcal{Z}^{(M+1)}_{M\,|\,i_1,\dots,i_{M+1}}(\varepsilon)\cdot U^{(0)}(\varepsilon)\,.$$

We get a similar conclusion: *$U(\underline{\kappa};\varepsilon)$ is perturbatively renormalizable at the order $M+1$ $(M=1,2,\dots)$* iff *the "divergent part" of the Mth order of the recursively constructed $U_M(\underline{\kappa},\varepsilon)$ is representable as a linear combination of first and zeroth order terms of the original $U(\underline{\kappa};\varepsilon)$.*

In the renormalization of cross sections in QFT we do not use any multiplicative renormalization as the physical quantity is a probability and is uniquely normalizable. In this case the sub-leading part in the perturbative expansion corresponds to a classical scattering. It is free of divergences and reflects the classical interaction Lagrangian. The renormalizability means that divergent parts at higher orders in perturbation theory under the above inductive process are representable as linear combinations of such classical cross sections.

Further problem that arises in perturbative renormalization is the preservation of additional structures during the renormalization process. For example, this can be preservation of a symmetry: we would like to construct a symmetry action of the type

$$F(g;\underline{\kappa})\cdot U^{\,\mathrm{ren}}(\underline{\kappa}) \;=\; U^{\,\mathrm{ren}}(\underline{f}(g;\underline{\kappa}))\,, \tag{5}$$

$$f_i(g;\underline{\kappa}) \;=\; \sum_{n=1}^{\infty}\;\sum_{i_1,\dots,i_n=1}^{N} f^{(n)}_{i;i_1,\dots,i_n}\,\kappa_{i_1}\,\cdots\,\kappa_{i_n}\,,$$

$$F(g;\underline{\kappa}) \;=\; \sum_{n=0}^{\infty}\;\sum_{i_1,\dots,i_n=1}^{N} F^{(n)}_{i_1,\dots,i_n}\,\kappa_{i_1}\,\cdots\,\kappa_{i_n}\,,$$

$$\underline{f}(g_1g_2;\underline{\kappa}) = \underline{f}(g_1;\underline{f}(g_2;\underline{\kappa}))\,,\quad F(g_1g_2;\underline{\kappa}) = F(g_1;\underline{f}(g_2;\underline{\kappa}))\cdot F(g_2;\underline{\kappa}), \tag{6}$$

for every g, g_1, g_2 belonging to the symmetry group $\mathfrak{G}$ of the physical model, where $\underline{f}(g;\underline{\kappa})$ is an action $\mathfrak{G}$ via *formal diffeomorphisms* on $\underline{\kappa}$, which keeps stable $\underline{\kappa}=0$ (that is why the sum in the expansion of $\underline{f}(g;\underline{\kappa})$ begins with $n=1$). The starting point is an action of $\mathfrak{G}$ up to order 1 in $\underline{\kappa}$.

The possibility to preserve the symmetry (or other structures) after the renormalization is generally a cohomological problem related to the ambiguity of the renormalization procedure. Let us consider this ambiguity. Note that, if $\underline{\mathcal{K}}(\underline{\kappa};\varepsilon)$ is a solution of the renormalization problem (2) and $\underline{X}(\underline{\kappa}) = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{N} \underline{X}^{(n)}_{i_1,\dots,i_n} \kappa_{i_1} \cdots \kappa_{i_n}$ is an arbitrary formal diffeomorphism then $\underline{\mathcal{K}}'(\underline{\kappa};\varepsilon) = \underline{\mathcal{K}}(\underline{X}(\underline{\kappa});\varepsilon)$ is again a solution. Similarly, in the case (3) with a multiplicative renormalization we can additionally apply some formal series with coefficients belonging to $\mathrm{End}(\mathfrak{V})$. We conclude that the renormalized series $U^{\mathrm{ren}}(\underline{\kappa})$ is defined, in general, up to a transformation of the form:

$$U^{\mathrm{ren}}(\underline{\kappa}) \mapsto M(\underline{\kappa}) \cdot U^{\mathrm{ren}}(\underline{X}(\underline{\kappa}))\,. \tag{7}$$

Thus, the renormalization ambiguity contains at least the group of formal diffeomorphisms. Let us describe this group. The group multiplication is the composition of formal power series. Assume we have two such series:

$$\underline{\kappa}' = \underline{X}(\underline{\kappa}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} \underline{X}^{(n)}_{i_1,\dots,i_n} \kappa_{i_1} \cdots \kappa_{i_n} \equiv \sum_{n=1}^{\infty} \frac{1}{n!}\, X^{(n)}(\underline{\kappa}^{\otimes n})\,,$$

$$\underline{\kappa}'' = \underline{Y}(\underline{\kappa}') = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} \underline{Y}^{(n)}_{i_1,\dots,i_n} \kappa'_{i_1} \cdots \kappa'_{i_n} \equiv \sum_{n=1}^{\infty} \frac{1}{n!}\, Y^{(n)}(\underline{\kappa}'^{\otimes n})\,,$$

whose coefficients $X^{(n)} = (\underline{X}^{(n)}_{i_1,\dots,i_n}) = (X^{(n)}_{i;i_1,\dots,i_n}) \in \mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\, \mathfrak{L})$ (and similarly $Y^{(n)} \in \mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\, \mathfrak{L})$) are described by *symmetric* n–linear maps on the vector space of couplings, $\mathfrak{L} = \mathbf{R}\kappa_1 \oplus \cdots \oplus \mathbf{R}\kappa_n$ ($S(\mathfrak{L}^{\otimes n})$ being the symmetrized tensor product). Note that we have introduced for convenience an extra $\frac{1}{n!}$ factors in the series $\underline{X}(\underline{\kappa})$. Then, the composition $\underline{Z}(\underline{\kappa}) = \underline{Y}(\underline{X}(\underline{\kappa}))$ is given by the formula ([2]):

$$Z^{(n)} = \sum_{\mathfrak{P} \in \mathrm{Part}\{1,\dots,n\}} Y^{(k)} \circ (X^{(j_1)} \otimes \cdots \otimes X^{(j_k)}) \circ \sigma_{\mathfrak{P}}\,, \tag{8}$$

where the sum is over the set $\mathrm{Part}\{1,\dots,n\}$ of all partitions of the set $\{1,\dots,n\}$ and the permutation $\sigma_{\mathfrak{P}} \in \mathcal{S}_n$ is defined by

$$\begin{aligned} \sigma_{\mathfrak{P}}^{-1} &:= (i_{1,1},\dots,i_{1,j_1},\dots,i_{k,1},\dots,i_{k,j_k})\,, \\ &\quad i_{\ell,1} < \cdots < i_{\ell,j_\ell}\,, \quad i_{1,1} < i_{2,2} < \cdots < i_{k,j_k}\,, \\ \text{for} \quad \mathfrak{P} &= \Big\{\{i_{1,1},\dots,i_{1,j_1}\},\dots,\{i_{k,1},\dots,i_{k,j_k}\}\Big\} \in \mathrm{Part}\{1,\dots,n\} \end{aligned}$$

(called *pointed unshuffle*) and $\sigma_{\mathfrak{P}}$ acts on $\mathfrak{L}^{\otimes n}$ in the obvious way (by permuting the factors). In the case $N = 1$ the above formula is known as

Faá-di-Bruno formula. We refer the reader to paper [2] for further information about the role of the *operads'* theory in the above construction, as well as, its further generalizations in renormalization theory.

Let us return to problem, stated in Eqs. (5) and (6), of extending symmetries on the renormalized perturbation series and describe in sketch the strategy of its solution. Our consideration will be in the spirit of the Gerstenhaber's theory of deformations of rings ([1]). Since the renormalization solution $U^{\,\mathrm{ren}}(\underline{\kappa})$ is defined up to a transformation of a form (7) we see that we can try to built the symmetry actions (6) so that they fulfill (5) for a given, already constructed, renormalized series $U^{\,\mathrm{ren}}(\underline{\kappa})$. The group actions up to the first order in $\underline{\kappa}$ are the starting point as we assumed in (4) that these orders are free of divergences and need not to be renormalized.

Let us set $f^{(n)}(g) : \underline{\kappa} \mapsto \underline{f}^{(n)}(g;\underline{\kappa}^{\otimes n})$, $f^{(n)}(g) \in \mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\mathfrak{L})$ for $n = 1,2,\dots$ and similarly, $F^{(n)}(g) : \underline{\kappa} \mapsto F^{(n)}(g;\underline{\kappa}^{\otimes n})$, $F^{(n)}(g) \in \mathrm{Hom}(S(\mathfrak{L}^{\otimes n}), \mathrm{End}(\mathfrak{V}))$ for $n = 0,1,\dots$ and note from Eq. (6) that its leading orders read

$$f^{(1)}(g_1g_2) \,=\, f^{(1)}(g_1) \circ f^{(1)}(g_2)\,, \quad F^{(0)}(g_1g_2) \,=\, F^{(0)}(g_1) \cdot F^{(0)}(g_2)\,, \tag{9}$$

$$F^{(1)}(g_1g_2) \,=\, F^{(0)}(g_1) \cdot F^{(1)}(g_2) + (F^{(1)}(g_1) \circ f^{(1)}(g_2)) \cdot F^{(0)}(g_2) \tag{10}$$

(where "$\cdot$" stands for the multiplication in $\mathrm{End}(\mathfrak{V})$ and "$\circ$" stands for the composition with maps belonging to $\mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\mathfrak{L})$ or $\mathrm{Hom}(S(\mathfrak{L}^{\otimes n}), \mathrm{End}(\mathfrak{V})))$. Equation (9) means that we have two representations of the symmetry group $\mathfrak{G}$: one on the vector space $\mathfrak{L}$ of couplings, $\mathfrak{G} \ni g \mapsto f^{(1)}(g) \in \mathrm{Aut}(\mathfrak{L})$, and one on the vector space $\mathfrak{V}$ where the physical quantity U takes values, $\mathfrak{G} \ni g \mapsto F^{(0)}(g) \in \mathrm{Aut}(\mathfrak{V})$. Similarly, the leading two orders in Eq. (5) read

$$\begin{aligned} &F^{(0)}(g) \cdot U^{\,\mathrm{ren},(0)} - U^{\,\mathrm{ren},(0)} \,=\, 0\,, \\ &F^{(0)}(g) \cdot U^{\,\mathrm{ren},(1)} + F^{(1)}(g) \cdot U^{\,\mathrm{ren},(0)} - U^{\,\mathrm{ren},(1)} \circ f^{(1)}(g) \,=\, 0\,. \end{aligned} \tag{11}$$

In particular, $U^{\,\mathrm{ren},(0)} \in \mathfrak{V}$ is a stable element for the action $F^{(0)}$. In general, if we fix the data in the first orders: $U^{\,\mathrm{ren},(k)} \equiv U^{(k)}$, for $k = 0,1$, $F^{(0)}$ and $f^{(1)}$ then we get linear inhomogeneous conditions for the order n data $U^{\,\mathrm{ren},(n)}$, $F^{(n)}$ and $f^{(n)}$:

$$\begin{aligned} &f^{(1)}(g_1) \circ f^{(n)}(g_2) - f^{(n)}(g_1g_2) + f^{(n)}(g_1) \circ f^{(1)}(g_2)^{\otimes n} \\ &\qquad\qquad = \text{ polynomial expression in } f^{(2)},\dots,f^{(n-1)}\,, \end{aligned} \tag{12}$$

$$\begin{aligned} &F^{(0)}(g_1) \cdot F^{(n)}(g_2) - F^{(n)}(g_1g_2) + (F^{(n)}(g_1) \circ f^{(1)}(g_2)^{\otimes n}) \cdot F^{(0)}(g_2) \\ &\qquad = \text{ polynomial expression in } f^{(2)},\dots,f^{(n-1)},F^{(1)},\dots,F^{(n-1)}\,, \end{aligned} \tag{13}$$

$$\begin{aligned} &F^{(0)}(g) \cdot U^{\,\mathrm{ren},(n)} + F^{(n)}(g) \cdot U^{\,\mathrm{ren},(0)} \\ &\qquad - U^{\,\mathrm{ren},(1)} \circ f^{(n)}(g) - U^{\,\mathrm{ren},(n)} \circ f^{(1)}(g)^{\otimes n} \end{aligned}$$

$$= \text{polynomial expression in } f^{(2)},\dots,f^{(n-1)},F^{(1)},\dots,F^{(n-1)}, \\ U^{\mathrm{ren},(2)},\dots,U^{\mathrm{ren},(n-1)}\,. \tag{14}$$

In the left hand sides of the above equations one can recognize certain cochain spaces of the group $\mathfrak{G}$. However, the right hand sides are highly nonlinear. To improve the latter nonlinearity one can pass from the symmetry group $\mathfrak{G}$ to its Lie algebra $\mathfrak{g}$. Then, in the left hand sides of the "infinitesimal" (i.e., Lie algebra) versions of the above identities we get Lie algebra cochain spaces of Chevalley–Eilenberg type. Each of these cochain spaces is a direct sum of three vector subspaces:

$$\begin{aligned}
\mathcal{C}^m_{1,n} &:= \mathrm{Maps}\Big(\mathfrak{g}^{\wedge m},\mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\mathfrak{L})\Big)\,,\\
\mathcal{C}^m_{2,n} &:= \mathrm{Maps}\Big(\mathfrak{g}^{\wedge m},\mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\mathrm{End}(\mathfrak{V}))\Big)\,,\\
\mathcal{C}^m_{3,n} &:= \mathrm{Maps}\Big(\mathfrak{g}^{\wedge (m-1)},\mathrm{Hom}(S(\mathfrak{L}^{\otimes n}),\mathfrak{V})\Big)\,,\\
\mathcal{C}^m_{12,n} &:= \mathcal{C}^m_{1,n}\oplus\mathcal{C}^m_{2,n}\,,\quad \mathcal{C}^m_{123,n} := \mathcal{C}^m_{12,n}\oplus\mathcal{C}^m_{3,n}\,.
\end{aligned}$$

Then, the right hand sides of Eqs. (12)–(14), taken on the Lie algebra $\mathfrak{g}$, define a differential

$$d:\mathcal{C}^1_{123,n}\to\mathcal{C}^2_{123,n}\,,\quad d(\mathcal{C}^1_{12,n})\subseteq\mathcal{C}^2_{12,n}\,,\quad d(\mathcal{C}^1_{1,n})\subseteq\mathcal{C}^2_{1,n}$$

and it can be extended to a differential of a cochain complex. Let us further introduce

$$\Xi = \Xi_{2,1}+\sum_{n=2}^{\infty}(\Xi_{1,n}+\Xi_{2,n}+\Xi_{3,n}) \in \bigoplus_{n=0}^{\infty}\Big(\mathcal{C}^m_{1,n}\oplus\mathcal{C}^m_{2,n}\oplus\mathcal{C}^m_{3,n}\Big)\,,$$

which combine the higher order data from the renormalization procedure:

$$\begin{aligned}
(\Xi_{1,n})_{n\geqslant 2} &= (f^{(2)},f^{(3)},\dots)\,,\\
(\Xi_{2,n})_{n\geqslant 1} &= (F^{(1)},F^{(2)},\dots)\,,\\
(\Xi_{3,n})_{n\geqslant 2} &= (U^{\mathrm{ren}(2)},U^{\mathrm{ren}(3)},\dots)\,.
\end{aligned}$$

Then, the Lie algebra version of Eqs. (12)–(14) take the form of a Maurer-Cartan equation:

$$d\,\Xi+\frac{1}{2}[\Xi,\Xi] = 0\,, \tag{15}$$

where the bracket is a Lie super-algebra bracket (Ξ being an odd element):

$$\begin{aligned}
&[\mathcal{C}^{m_1}_{1,n_1},\mathcal{C}^{m_2}_{1,n_2}]\subseteq\mathcal{C}^{m_1+m_2}_{1,n_1+n_2-1}\,, && [\mathcal{C}^{m_1}_{2,n_1},\mathcal{C}^{m_2}_{2,n_2}]\subseteq\mathcal{C}^{m_1+m_2}_{2,n_1+n_2}\,,\\
&[\mathcal{C}^{m_1}_{1,n_1},\mathcal{C}^{m_2}_{2,n_2}]\subseteq\mathcal{C}^{m_1+m_2}_{2,n_1+n_2-1}\,, && [\mathcal{C}^{m_1}_{1,n_1},\mathcal{C}^{m_2}_{3,n_2}]\subseteq\mathcal{C}^{m_1+m_2}_{3,n_1+n_2-1}\,,\\
&[\mathcal{C}^{m_1}_{2,n_1},\mathcal{C}^{m_2}_{3,n_2}]\subseteq\mathcal{C}^{m_1+m_2}_{3,n_1+n_2}\,, && [\mathcal{C}^{m_1}_{3,n_1},\mathcal{C}^{m_2}_{3,n_2}] = 0\,.
\end{aligned}$$

Thus, Eq. (15) can be solved inductively in $n = 2, 3, \ldots$ as its right hand side, due to the above grading properties of the bracket, will depend on the lower orders. The existence of solutions of these equations is controlled by the second cohomology groups of the above complex and the uniqueness of the solution is modulo exact corrections.

References

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