Symmetries and solutions of field equations of axion electrodynamics

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Abstract

The group classification of models of axion electrodynamics with arbitrary self interaction of axionic field is carried out. It is shown that extensions of the basic Poincaré invariance of these models appear only for constant and exponential interactions. The related conservation laws are discussed. Exact solutions for the electromagnetic and axion fields are discussed including those ones which describe propagation with group velocities faster than the speed of light. However these solutions are causal since the corresponding energy velocities are subluminal.

1. Introduction

The group analysis of PDEs is a fundamental field including many interesting internal problems. But maybe the most attractive feature of the group analysis is its great value for various applications such as defining of maximal Lie symmetries of complicated physical models, construction of models with a priory requested symmetries, etc. Sometimes the group analysis is the only way to find exact solutions for nonlinear problems.

In the present paper we make the group classification of the field equations of axion electrodynamics with arbitrary self interaction of axion field. The considered model includes the standard axion electrodynamics as a particular case. We prove that an extension of the basic Poincaré invariance appears only for the exponential, constant and trivial interaction terms.

Let us present physical motivations of this research. To explain the absence of the CP symmetry violation in interquark interactions Peccei and Quinn [1] suggested that a new symmetry must be present. The breakdown of this gives rise to the axion field proposed later by Weinberg [2] and Wilczek [3]. And it was Wilczek who presented the first analysis of possible effects caused by axions in electrodynamics [4].

Axions belong to the main candidates to form the dark matter, see, e.g., [5] and references cited therein. New arguments for the materiality of axion

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theories were created in solid states physics. Namely, it was found recently [6] that the axionic-type interaction terms appear in the theoretical description of a class of crystalline solids called topological insulators. In other words, although their existence is still not confirmed experimentally axions are stipulated at least in the three fundamental fields: QCD, cosmology and condensed matter physics. That is why we decide to make group analysis of axionic theories and find in some sense completed set of the related exact solutions.

2. Field equations of axion electrodynamics

We start with the following model Lagrangian:

$$L = \frac{1}{2}p_\mu p^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu} - V(\theta). \tag{1}$$

Here $F_{\mu\nu}$ is the strength tensor of electromagnetic field, $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $p_\mu = \partial_\mu\theta$, θ is the pseudoscalar axion field, $V(\theta)$ is a function of θ, κ is a dimensionless constant, and the summation is imposed over the repeating indices over the values 0, 1, 2, 3. Moreover, the strength tensor can be expressed via the four-potential $A = (A^0, A^1, A^2, A^3)$ as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2}$$

Setting in (1) $\theta = 0$ we obtain the Lagrangian for Maxwell field. Moreover, if θ is a constant then (1) coincides with the Maxwell Lagrangian up to constant and four-divergence terms. Finally, the choice $V(\theta) = \frac{1}{2}m^2\theta^2$ reduces L to the standard Lagrangian of axion electrodynamics.

We will investigate symmetries of the generalized Lagrangian (1) with arbitrary $V(\theta)$. More exactly, we will make the group classification of the corresponding Euler-Lagrange equations:

$$\begin{aligned}
&\nabla\cdot\mathbf{E} = \kappa\mathbf{p}\cdot\mathbf{B},\\
&\partial_0\mathbf{E} - \nabla\times\mathbf{B} = \kappa(p_0\mathbf{B} + \mathbf{p}\times\mathbf{E}), \qquad (3)\\
&\nabla\cdot\mathbf{B} = 0, \quad \partial_0\mathbf{B} + \nabla\times\mathbf{E} = 0,\\
&\Box\theta = -\kappa\mathbf{E}\cdot\mathbf{B} + F, \qquad (4)
\end{aligned}$$

where

$$\mathbf{B} = \{B^1, B^2, B^3\}, \quad \mathbf{E} = \{E^1, E^2, E^3\}, \quad E^a = F^{0a}, \quad B^a = \frac{1}{2}\varepsilon^{abc}F_{bc},$$

$$F = \frac{\partial\varphi}{\partial\theta}, \quad \Box = \partial_0^2 - \nabla^2, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad p_0 = \frac{\partial\theta}{\partial x_0}, \quad \mathbf{p} = \nabla\theta.$$

3. Group classification of systems (3)–(4)

Equations (3)–(4) include an arbitrary function $F(\theta)$ so we can expect that the variety of symmetries of this system depends on the explicit form of F. The group classification of these equations presupposes finding their symmetry groups for arbitrary F.

The maximal continuous symmetry of system (3)–(4) with arbitrary function $F(\theta)$ is given by Poincaré group $P(1,3)$. Infinitesimal generators of this group take the following form:

$$\begin{aligned}
&P_0 = \partial_0, \quad P_a = \partial_a, \\
&J_{ab} = x_a\partial_b - x_b\partial_a + B^a\partial_{B^b} - B^b\partial_{B^a} + E^a\partial_{E^b} - E^b\partial_{E^a}, \\
&J_{0a} = x_0\partial_a + x_a\partial_0 + \varepsilon_{abc}\left(E^b\partial_{B^c} - B^b\partial_{E^c}\right),
\end{aligned} \tag{5}$$

where ε_{abc} is the unit antisymmetric tensor, $a, b, c = 1, 2, 3$.

Operators (5) form a basis of the Lie algebra p(1,3) of the Poincaré group.

For some special functions $F(\theta)$ symmetry of system (3)–(4) appears to be more extended. Namely, if $F = 0$, $F = c$ or $F = be^{a\theta}$ then the basis (5) of symmetry algebra of this system is extended by the following additional operators P_4, D and X:

$$\begin{aligned}
&P_4 = \partial_\theta,\ D = x_0\partial_0 + x_i\partial_i - \frac{1}{2}F^{\mu\nu}\partial_{F^{\mu\nu}} \quad \text{if} \quad F(\theta) = 0, \\
&P_4 = \partial_\theta \quad \text{if} \quad F(\theta) = c, \\
&X = aD - 2P_4 \quad \text{if} \quad F(\theta) = be^{a\theta}.
\end{aligned} \tag{6}$$

Operator P_4 generates shifts of dependent variable θ, D is the dilatation operator generating a consistent scaling of dependent and independent variables, and X generates the simultaneous shift and scaling. Note that arbitrary parameters a, b and c can be reduced to the fixed values $a = \pm 1$, $b = \pm 1$ and $c = \pm 1$ by scaling dependent and independent variables.

Thus the continues symmetries of system (3)–(4) where $F(\theta)$ is an arbitrary function of θ are exhausted by the Poincaré group. The same symmetry is accepted by the standard equations of axion electrodynamics which correspond to $F(\theta) = -m^2\theta$. In the cases indicated in (6) we have the extended Poincaré groups.

4. Conservation laws

An immediate consequence of symmetries presented above is the existence of conservation laws. Indeed, the system (3)–(4) admits a Lagrangian formulation. Thus, in accordance with the Noether theorem, symmetries of equations (3)–(4) which keep the shape of Lagrangian (1) up to four divergence terms should generate conservation laws. Let us present them explicitly.

First we represent generators (5) and (6) written in terms of the variational variables A^μ and $A^4 = \theta$ in the following unified form

$$Q = \xi^\mu \partial_\mu + \varphi^\tau \partial_{A^\tau}, \tag{7}$$

where the summation is imposed over the values $\tau = 0, 1, 2, 3, 4$ and $\mu = 0, 1, 2, 3$.

Conserved current corresponding to symmetry (7) can be represented as [7]:

$$J_\sigma = \varphi_\tau \frac{\partial L}{\partial(\partial_\sigma A_\tau)} + \xi^\sigma L - \xi^\nu \partial_\nu A^\tau \frac{\partial L}{\partial(\partial_\sigma A^\tau)}. \tag{8}$$

The basic conserved quantity is the energy momentum tensor which corresponds to symmetries P_μ presented in (5). In this case $\varphi^\tau \equiv 0$ and $\xi_\mu = 1$. Starting with (1) and using three dimensional notations

$$F_{0a} = E_a, \quad F_{ab} = \varepsilon_{abc} B_c, \tag{9}$$

we find the conserved energy momenta tensor in the following form

$$\begin{aligned}
T^{00} &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 + p_0^2 + \mathbf{p}^2) + V(\theta), \\
T^{0a} &= T^{a0} = \varepsilon_{abc} E_b B_c + p^0 p^a, \\
T^{ab} &= -E^a E^b - B^a B^b + p^a p^b + \frac{1}{2}\delta^{ab}(\mathbf{E}^2 + \mathbf{B}^2 + p_0^2 - \mathbf{p}^2 - 2V(\theta)).
\end{aligned} \tag{10}$$

The tensor $T^{\mu\nu}$ is symmetric and satisfies the continuity equation $\partial_\nu T^{\mu\nu} = 0$. Its components T^{00} and T^{0a} are associated with the energy and momentum densities.

It is important to note that the energy momentum tensor does not depend on parameter κ and so it is not affected by the term $\frac{\kappa}{4}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu}$ present in Lagrangian (1). In fact this tensor is nothing but a sum of energy momenta tensors for the free electromagnetic field and scalar field. Moreover, the interaction of these fields between themselves is not represented in (10).

5. Selected exact solutions

The field equations of axion electrodynamics form a rather complicated system of nonlinear partial differential equations. However, this system admits the extended symmetry algebra, i.e., p(1,3), which makes it possible to find a number of exact solutions. Here we present some of these solutions while the completed list of them can be found in [8].

The algorithm for construction of group solutions of partial differential equations goes back to Sophus Lie and is expounded in various monographs, see, e.g., [7]. Roughly speaking, to find such solutions we have to change the dependent and independent variables by invariants of the subgroups of our equations symmetry group. Solving equations (3)–(4) it is reasonable to restrict ourself to three-parametrical subgroups which make it possible to reduce (3)–(4) to systems of *ordinary* differential equations. The complete list of these subgroups can be found in [9].

To make solutions of equations (3)–(4) more physically transparent, we write them in terms of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ whose components are expressed via the strengths tensor $F_{\mu\nu}$ as shown in (9). In addition, we rescale the dependent variables such that $\kappa \to 1$.

5.1. Plane wave solutions

Let us present solutions of system (3)–(4) which are invariant w.r.t. subalgebras of p(1,3) whose basis elements have the following unified form: $\langle P_1, P_2, kP_0 + \varepsilon P_3 \rangle$ where ε and k are parameters satisfying $\varepsilon^2 \neq k^2$, while $P_1,\ P_2,\ P_3$ and P_0 are generators given in (1).

The invariants ω of the corresponding three-parametrical group should solve the equations

$$P_1\omega = 0, \quad P_2\omega = 0, \quad (kP_0 + \varepsilon P_3)\omega = 0. \tag{11}$$

Solutions of (11) include all dependent variables E_a, B_a, θ $(a = 1, 2, 3)$ and the only independent variable $\omega = \varepsilon x_0 - k x_3$. Thus we can search for solutions which are functions of ω only. As a result we reduce equations (3)–(4), (9) to the system of ordinary differential equations whose solutions are:

$$\begin{aligned} &B_1 = -kc_1\theta, \quad B_2 = \varepsilon c_1 + kc_2, \quad B_3 = c_3, \\ &E_1 = \varepsilon c_2 + kc_1, \quad E_2 = \varepsilon c_1\theta, \quad E_3 = c_3\theta - c_4(\varepsilon^2 - k^2), \end{aligned} \tag{12}$$

where $c_1, ..., c_4$ are arbitrary real numbers. The corresponding bounded solutions of equation (4) with $F = -m^2\theta$ are

$$\theta = a_\mu \cos\mu\omega + r_\mu \sin\mu\omega + \frac{c_3c_4}{\mu^2}, \tag{13}$$

where a_μ, r_μ and μ are arbitrary constants restricted by the following constraint

$$\mu^2 = \left(c_1^2 + \frac{c_3^2 + m^2}{\varepsilon^2 - k^2}\right). \tag{14}$$

Notice that for the simplest non-linear function $F = \lambda\theta^2$ equation (4) is reduced to Weierstrass equation and admits a nice soliton-like solution

$$\theta = \frac{c_3c_4}{2}\tanh^2(\omega + C), \tag{15}$$

where C is an integration constant. The related parameters ε, k and λ should satisfy the conditions

$$\varepsilon^2 = k^2 + \frac{c_3^2}{8 - c_1^2}, \quad \lambda c_3 c_4 = 12. \tag{16}$$

The corresponding magnetic, electric and axion fields are localized waves moving along the third coordinate axis.

In analogous (but as a rule much more complicated) way we can find solutions corresponding to the other three-dimensional subalgebras of the Poincaré algebra. One more and rather specific solution of equations (3)–(4) with $\kappa = 1$ and $F = 0$ (obtained with using the subalgebra spanned on basis elements $\langle J_{12} + kP_0 + \varepsilon P_1, P_2, P_3\rangle$) can be written as follows:

$$\begin{aligned} &E_1 = \varepsilon(c_k \sin(\omega) - d_k \cos(\omega)), \quad E_2 = \varepsilon(c_k \cos(\omega) + d_k \sin(\omega)), \\ &E_3 = e, \quad B_1 = -\tfrac{k}{\varepsilon}E_2, \quad B_2 = \tfrac{k}{\varepsilon}E_1, \quad B_3 = 0, \quad \theta = \alpha x_0 + \nu x_3 + \mu, \end{aligned} \tag{17}$$

where $e, c_k, d_k, \varepsilon, k, \alpha, \nu, \mu$ are constants satisfying the following conditions

$$\varepsilon^2 - k^2 = \nu\varepsilon - \alpha k, \quad \varepsilon \neq 0. \tag{18}$$

Solutions (17) depend on two different plane wave variables, i.e., $\omega = \varepsilon x_0 - kx_1$ and $\alpha x_0 + \nu x_1$. They satisfy the superposition principle since a sum of solutions with different ε, k, c_k and d_k is also a solution of equations (3)–(4) with $\kappa = 1$ and $F = 0$. Thus it is possible to generate much more general solutions by summing up functions (17) over k and treating c_k and d_k as arbitrary functions of k.

5.2. Radial and planar solutions

Consider solutions which include the Coulomb electric field. They can be obtained using invariants of the subalgebra spanned on $\langle J_{12},\ J_{23},\ J_{31}\rangle$ and have the following form

$$B_a = \frac{c_1 x_a}{r^3}, \quad E_a = \frac{(c_1\theta - c_2)x_a}{r^3}, \quad \theta = \frac{\varphi}{r}, \tag{19}$$

where φ is a function of x_0 and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ satisfying the following equation

$$\frac{\partial^2\varphi}{\partial r^2} - \frac{\partial^2\varphi}{\partial x_0^2} = \left(\frac{c_1^2}{r^4} + m^2\right)\varphi - \frac{c_1 c_2}{r^3}. \tag{20}$$

Setting in (19) $c_1 = 0$ we come to the electric field of point charge which is well defined for $r > 0$. A particular solution for (20) corresponding to

$c_1 = -q^2 < 0$ and $c_2 = 0$ is $\varphi = c_3 r \sin(mx_0)e^{-\frac{q^2}{r}}$ which gives rise to the following field components

$$B_a = -\frac{q^2 x_a}{r^3}, \quad E_a = -\frac{q^2 \theta x_a}{r^3}, \quad \theta = c_3 \sin(mx_0)e^{-\frac{q^2}{r}}. \tag{21}$$

The components of magnetic field B_a are singular at $r = 0$ while E_a and θ are bounded for $0 \leq r \leq \infty$.

Separating variables it is possible to find the general solution of equation (20), see [8].

One more solution of equations (3)–(4) for $F = 0$ with a radial electric field is

$$E_a = \frac{x_a}{r^2}. \tag{22}$$

The corresponding magnetic and axion fields take the following forms

$$B_1 = \frac{x_1 x_3}{r^2 x}, \quad B_2 = \frac{x_2 x_3}{r^2 x}, \quad B_3 = -\frac{x}{r^2}, \quad \theta = \arctan\left(\frac{x}{x_3}\right),$$

where $x = \sqrt{x_1^2 + x_2^2}$.

The electric field (22) is requested in the superintegrable model with Fock symmetry proposed in [10].

Let us present planar solutions which depend on spatial variables x_1 and x_2. Namely, the functions

$$\begin{aligned} &E_1 = x_1\left(c_1 x^{c_3-2} + c_2 x^{-2-c_3}\right), \quad E_2 = x_2\left(c_1 x^{c_3-2} + c_2 x^{-2-c_3}\right), \\ &B_1 = x_2\left(c_1 x^{c_3-2} - c_2 x^{-2-c_3}\right), \quad B_2 = x_1\left(c_2 x^{-2-c_3} - c_1 x^{c_3-2}\right), \\ &E_3 = 0, \quad B_3 = 0, \quad \theta = c_3 \arctan\frac{x_2}{x_1} + c_4, \end{aligned} \tag{23}$$

where $c_1, ..., c_4$ are arbitrary parameters, solve equations (3)–(4) with $\kappa = 1$ and $F = 0$.

Solutions (23) can be found with using invariants of a subgroup of the *extended* Poincaré group whose Lie algebra is spanned on the basis $\langle P_0,\ P_3,\ J_{12} + P_4\rangle$, see equations (5), (6) for definitions.

6. Phase, group and energy velocities

In this section we consider some of found solutions in more details and discuss the propagation velocities of the corresponding fields. There are various notions of field velocities, see, e.g., [11, 12, 13]. We shall discuss the phase, group and energy velocities.

Let us start with the plane wave solutions given by equations (12) and (13). They describe oscillating waves moving along the third coordinate axis. Setting for simplicity $c_2 = c_3 = c_4 = r_\mu = 0$ we obtain

$$\begin{aligned} &B_1 = c_1 k\theta, \quad B_2 = -c_1\varepsilon, \quad B_3 = 0, \quad E_1 = -c_1 k, \\ &E_2 = -c_1\varepsilon\theta, \quad E_3 = 0, \quad \theta = a_\mu \cos(\mu(\varepsilon x_0 - k x_3)). \end{aligned} \tag{24}$$

Here ε, k, and a_μ are arbitrary parameters which, in accordance with (14), should satisfy the following dispersion relations

$$(\varepsilon^2 - k^2)(\mu^2 - c_1^2) = m^2. \tag{25}$$

If $m \neq 0$ the version $\mu^2 = c_1^2$ is forbidden, and we have two qualitatively different possibilities: $\mu^2 > c_1^2$ and $\mu^2 < c_1^2$.

Let $\mu^2 > c_1^2$ then $(\varepsilon^2 - k^2) = \frac{m^2}{\mu^2 - c_1^2} > 0$. The corresponding group velocity V_g is equal to the derivation of ε w.r.t. k, i.e.,

$$V_g = \frac{\partial\varepsilon}{\partial k} = \frac{k}{\varepsilon}\,. \tag{26}$$

Since $\varepsilon > k$, the group velocity appears to be less than the velocity of light (remember that we use the Heaviside units in which the velocity of light is equal to 1).

On the other hand the phase velocity $V_p = \frac{\varepsilon}{k}$ is larger than the velocity of light, but this situation is rather typical in relativistic field theories.

In the case $\mu^2 < c_1^2$ the wave number k is larger than ε. As a result the group velocity (26) exceeds the velocity of light, and we have a phenomenon of superluminal motion. To understand wether the considered solutions are causal let us calculate the energy velocity which is equal to the momentum density divided by the energy density

$$V_e = \frac{T^{03}}{T^{00}}\,. \tag{27}$$

Substituting (24) into (10) we find the expressions for T^{00} and T^{03}:

$$T^{00} = \frac{1}{2}(\varepsilon^2 + k^2)\Phi + \frac{1}{2}m^2\theta^2, \quad T^{03} = \varepsilon k\Phi,$$

where $\Phi = c_1^2(\theta^2 + 1) + \mu^2(a_\mu^2 - \theta^2)$. Thus

$$V_e = \frac{2\varepsilon k\Phi}{(\varepsilon^2 + k^2)\Phi + \frac{1}{2}m^2\theta^2} < \frac{2\varepsilon k}{\varepsilon^2 + k^2} < 1, \tag{28}$$

and this relation is valid for $\varepsilon > k$ and for $\varepsilon < k$ as well.

We see that the energy velocity is less than the velocity of light. Thus solutions (24) can be treated as causal in spite of the fact that for $\mu^2 < c_1^2$ the group velocity is superluminal.

7. Conclusions

We have performed group classification of field equations of axion electrodynamics (3)–(4) which include an arbitrary function F depending on θ, and have found the conservation laws generated by these equations. Exact solutions corresponding to three-dimensional subalgebras of the Poincaré algebra have been found and presented in [8]. There are 32 types of such solutions defined up to arbitrary constants or arbitrary functions. Some of these solutions can have interesting applications, e.g., for construction of exactly solvable problems for Dirac fermions. Solutions describing the faster-than-light propagation are admissible. However, these solutions are causal since the corresponding energy velocity is subluminal.

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