Bosonization of Superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level *

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Abstract

We give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. The bosonization of level $k \in \mathbf{C}$ is completely different from those of level $k = 1$. From this bosonization, we induce the Wakimoto realization whose character coincides with those of the Verma module. We give the screening that commute with $U_q(\widehat{sl}(N|1))$. Using this screening, we propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. We study non-vanishing property of the correlation function defined by a trace of the vertex operators.

1. Introduction

Bosonizations provide a powerful method to construct correlation function of exactly solvable models. We construct a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ $(N \geq 2)$ for an arbitrary level $k \in \mathbf{C}$ [1, 2]. For the special level $k = 1$, bosonizations have been constructed for the quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3, 4, 5, 6, 7, 8, 9, 10]. Bosonizations of level $k \in \mathbf{C}$ are completely different from those of level $k = 1$. For an arbitrary level $k \in \mathbf{C}$ bosonizations have been studied only for $U_q(\widehat{sl}_N)$ [11, 12] and $U_q(\widehat{sl}(N|1))$ [1, 2]. Our construction is based on the ghost-boson system. We need more consideration to get the Wakimoto realization whose character coincides with those of the Verma module. Using ξ-η system we construct the Wakimoto realization [13, 14] from our level k bosonization. For an arbitrary level $k \neq -N + 1$ we construct the screening current that commutes with $U_q(\widehat{sl}(N|1))$ modulo total difference. By using Jackson integral and the screening current, we construct the screening that commute with

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$U_q(\widehat{sl}(N|1))$ [13, 15]. We propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. By using the Gelfand-Zetlin basis, we have checked the intertwining property of the vertex operator for rank $N = 2, 3, 4$ [15]. We balance the background charge of the vertex operator by using the screening and propose the correlation function by a trace of them, which gives quantum and super generalization of Dotsenko-Fateev theory [16].

The paper is organized as follows. In section 2 we review bosonizations of $U_q(\widehat{sl}_2)$. In section 3 we construct a bosonization of $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. We induce the Wakimoto realization by ξ-η system. In section 4 we construct the screening that commute with $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N+1$. We propose the vertex operator and the correlation function.

2. Bosonization : Level $k = 1$ vs. Level $k \in \mathbf{C}$

In this section we review the bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$. The purpose of this section is to make readers understand that the bosonization of level $k \in \mathbf{C}$ is complete different from those of level $k = 1$. In what follows let q be a generic complex number $0 < |q| < 1$. We use the standard q-integer notation :

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

First we recall the definition of $U_q(\widehat{sl}_2)$. We recall the Drinfeld realization of the quantum affine algebra $U_q(\widehat{sl}_2)$.

Definition 2.1 *[17] The generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ are $x^{\pm}_{i,n}$, h_m, h, c $(n \in \mathbf{Z}, m \in \mathbf{Z}_{\neq 0})$. Defining relations are*

$$\begin{aligned}
&c : \text{central},\ \ [h, h_m] = 0,\\
&[h_m, h_n] = \delta_{m+n,0}\frac{[2m]_q[cm]_q}{m},\\
&[h, x^{\pm}(z)] = \pm 2x^{\pm}(z),\\
&[h_m, x^{\pm}(z)] = \pm\frac{[2m]_q}{m}q^{\mp\frac{c|m|}{2}}z^m x^{\pm}(z),\\
&(z_1 - q^{\pm 2}z_2)x^{\pm}(z_1)x^{\pm}(z_2) = (q^{\pm 2}z_1 - z_2)x^{\pm}(z_2)x^{\pm}(z_1),\\
&[x^+(z_1), x^-(z_2)] = \frac{1}{(q-q^{-1})z_1z_2}\\
&\qquad\qquad \times\left(\delta(q^{-c}z_1/z_2)\psi^+(q^{\frac{c}{2}}z_2) - \delta(q^c z_1/z_2)\psi^-(q^{-\frac{c}{2}}z_2)\right).
\end{aligned}$$

where we have used $\delta(z)=\sum_{n\in\mathbf{Z}} z^n$. *We have set the generating function*

$$\begin{aligned} x^\pm(z) &= \sum_{n\in\mathbf{Z}} x_n^\pm z^{-n-1}, \\ \psi^\pm(q^{\pm\frac{c}{2}}z) &= q^{\pm h} e^{\pm(q-q^{-1})\sum_{m>0} h_{\pm m} z^{\mp m}}. \end{aligned}$$

When the center c takes the complex number $c=k\in\mathbf{C}$, we call it the level k representation. We call the realization by the differential operators the bosonization. Frenkel-Jing [3] constructed the level $k=1$ bosonization of the quantum affine algebra $U_q(g)$ for simply-laced $g=(ADE)^{(1)}$. Here we recall the level $k=1$ bosonization of $U_q(\widehat{sl}_2)$. We introduce the boson a_n $(n\in\mathbf{Z}_{\neq 0})$ and the zero-mode operator ∂, α by

$$[a_m, a_n] = \frac{[2m]_q [m]_q}{m}\,\delta_{m+n,0}, \qquad [\partial, \alpha] = 2.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.2 *[3] A bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ for the level $k=1$ is given as follows.*

$$\begin{aligned} &c=1, \quad h=\partial, \quad h_n = a_n, \\ &x^\pm(z) =: e^{\mp\sum_{n\neq 0}\frac{a_n}{[n]_q} q^{\mp\frac{n}{2}} z^{-n} \pm(\alpha+\partial)} :. \end{aligned}$$

We have used the normal ordering symbol $::$

$$: a_k a_l := \begin{cases} a_k a_l & (k<0), \\ a_l a_k & (k>0), \end{cases} \qquad : \alpha\partial :=: \partial\alpha := \alpha\partial.$$

Next we recall the level k bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ [11]. We introduce the bosons and the zero-mode operator $a_n, b_n, c_n, Q_a, Q_b, Q_c$ $(n\in\mathbf{Z})$ as follows.

$$\begin{aligned} [a_m, a_n] &= \delta_{m+n,0}\frac{[2m]_q[(k+2)m]_q}{m}, & [\tilde{a}_0, Q_a] &= 2(k+2), \\ [b_m, b_n] &= -\delta_{m+n,0}\frac{[2m]_q[2m]_q}{m}, & [\tilde{b}_0, Q_b] &= -4, \\ [c_m, c_n] &= \delta_{m+n}\frac{[2m]_q[2m]_q}{m}, & [\tilde{c}_0, Q_c] &= 4, \end{aligned}$$

where $\tilde{a}_0=\frac{q-q^{-1}}{2\log q}a_0$, $\tilde{b}_0=\frac{q-q^{-1}}{2\log q}b_0$, $\tilde{c}_0=\frac{q-q^{-1}}{2\log q}c_0$. It is convenient to introduce the generating function $a(N|z;\alpha)$.

$$a(N|z;\alpha) = -\sum_{n\neq 0}\frac{a_n}{[Nn]_q} q^{|n|\alpha} z^{-n} + \frac{\tilde{a}_0}{N}\log z + \frac{Q_a}{N}.$$

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

Theorem 2.3 *[11] A bosonization of the quantum affine algebra $U_q(\widehat{sl}_2)$ for the level $k \in \mathbf{C}$ is given as follows.*

$$c = k \in \mathbf{C}, \quad h = a_0 + b_0,$$
$$h_m = q^{2m-|m|} a_m + q^{(k+2)m - \frac{k+2}{2}|m|} b_m,$$
$$x^+(z) = \frac{-1}{(q-q^{-1})z} \Big(: e^{-b(2|q^{-k-2}z;1) - c(2|q^{-k-1}z;0)} : \\ : e^{-b(2|q^{-k-2}z;1) - c(2|q^{-k-3}z;0)} : \Big),$$
$$x^-(z) = \frac{1}{(q-q^{-1})z} \Big(: e^{a(k+2|q^k z, -\frac{k+2}{2}) - a(k+2|q^{-2}z; \frac{k+2}{2}) + b(2|z;-1) + c(2|q^{-1}z;0)} : \\ - : e^{a(k+2|q^{-k-4}z; -\frac{k+2}{2}) - a(k+2|q^{-2}z; \frac{k+2}{2}) + b(2|q^{-2k-4}z;-1) + c(2|q^{-2k-3}z;0)} : \Big).$$

The level $k = 1$ bosonization is given by "monomial". The level $k \in \mathbf{C}$ bosonization is given by "sum". They are completely different.

3. Bosonization of Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we study the bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$.

3.1. Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we recall the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We fix a generic complex number q such that $0 < |q| < 1$. The Cartan matrix $(A_{i,j})_{0 \le i,j \le N}$ of the affine Lie algebra $\widehat{sl}(N|1)$ is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i \delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.$$

Here we set $\nu_1 = \cdots = \nu_N = +, \nu_{N+1} = \nu_0 = -$. We introduce the orthonormal basis $\{\epsilon_i | i = 1, 2, \cdots, N+1\}$ with the bilinear form, $(\epsilon_i|\epsilon_j) = \nu_i \delta_{i,j}$. Define $\bar{\epsilon}_i = \epsilon_i - \frac{\nu_i}{N-1}\sum_{j=1}^{N+1} \epsilon_j$. Note that $\sum_{j=1}^{N} \bar{\epsilon}_j = 0$. The classical simple roots $\bar{\alpha}_i$ and the classical fundamental weights $\bar{\Lambda}_i$ are defined by $\bar{\alpha}_i = \nu_i \epsilon_i - \nu_{i+1}\epsilon_{i+1}$, $\bar{\Lambda}_i = \sum_{j=1}^{i} \bar{\epsilon}_j$ $(1 \le i \le N)$. Introduce the affine weight Λ_0 and the null root δ satisfying $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\delta) = 1$, $(\Lambda_0|\epsilon_i) = 0$, $(\delta|\epsilon_i) = 0$, $(1 \le i \le N)$. The other affine weights and the affine roots are given by $\alpha_0 = \delta - \sum_{j=1}^{N} \bar{\alpha}_j$, $\alpha_i = \bar{\alpha}_i$, $\Lambda_i = \bar{\Lambda}_i + \Lambda_0$, $(1 \le i \le N)$. Let $P = \oplus_{j=1}^{N} \mathbf{Z}\Lambda_j \oplus \mathbf{Z}\delta$ and $P^* = \oplus_{j=1}^{N} \mathbf{Z}h_j \oplus \mathbf{Z}d$ the affine $\widehat{sl}(N|1)$ weight lattice and its dual lattice, respectively.

Definition 3.1 *[18] The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ are*

generated by the generators h_i, e_i, f_i $(0 \leq i \leq N)$. *The* $\mathbf{Z}_2$*-grading of the generators are* $|e_0| = |f_0| = |e_N| = |f_N| = 1$ *and zero otherwise. The defining relations are given as follows.*
The Cartan-Kac relations: *For* $N \geq 2$, $0 \leq i, j \leq N$, *the generators subject to the following relations.*

$$[h_i, h_j] = 0, \ [h_i, e_j] = A_{i,j} e_j, \ [h_i, f_j] = -A_{i,j} f_j, \ [e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$

The Serre relations: *For* $N \geq 2$, *the generators subject to the following relations for* $1 \leq i \leq N-1$, $0 \leq j \leq N$ *such that* $|A_{i,j}| = 1$.

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, \qquad [f_i, [f_i, f_j]_{q^{-1}}]_q = 0.$$

For $N \geq 2$, *the generators subject to the following relations for* $0 \leq i, j \leq N$ *such that* $|A_{i,j}| = 0$.

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0.$$

For $N \geq 3$, *the Serre relations of fourth degree hold.*

$$[e_N, [e_0, [e_N, e_{N-1}]_{q^{-1}}]_q] = 0, \quad [e_0, [e_1, [e_0, e_N]_q]_{q^{-1}}] = 0,$$
$$[f_N, [f_0, [f_N, f_{N-1}]_{q^{-1}}]_q] = 0, \quad [f_0, [f_1, [f_0, f_N]_q]_{q^{-1}}] = 0.$$

For $N = 2$, *the extra Serre relations of fifth degree hold.*

$$[e_2, [e_0, [e_2, [e_0, e_1]_q]]]_{q^{-1}} = [e_0, [e_2, [e_0, [e_2, e_1]_q]]]_{q^{-1}},$$
$$[f_2, [f_0, [f_2, [f_0, f_1]_q]]]_{q^{-1}} = [f_0, [f_2, [f_0, [f_2, f_1]_q]]]_{q^{-1}}.$$

Here and throughout this paper, we use the notations

$$[X, Y]_\xi = XY - (-1)^{|X||Y|} \xi YX.$$

We write $[X, Y]_1$ *as* $[X, Y]$ *for simplicity.*
The quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ has the $\mathbf{Z}_2$-graded Hopf-algebra structure. We take the following coproduct

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i,$$
$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$

and the antipode

$$S(e_i) = -q^{-h_i} e_i, \qquad S(f_i) = -f_i q^{h_i}, \qquad S(h_i) = -h_i.$$

The coproduct Δ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\Delta(Y)$ and the antipode S satisfies a $\mathbf{Z}_2$-graded algebra anti-automorphism $S(XY) = (-1)^{|X||Y|} S(Y)S(X)$. The multiplication rule for the tensor product is $\mathbf{Z}_2$-graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\widehat{sl}(N|1))$ and $v \in V, w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{|Y||X'|} XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{|Y||v|} Xv \otimes Yw$, which extends to inhomogeneous elements through linearity.

Definition 3.2 *The quantum superalgebra $U_q(\widehat{sl}(N|1))$ is the subalgebra of $U_q(\widehat{sl}(N|1))$, that is generated by $e_1, e_2, \cdots, e_N$, $f_1, f_2, \cdots, f_N$, and $h_1, h_2, \cdots, h_N$.*

We recall the Drinfeld realization of $U_q(\widehat{sl}(N|1))$, that is convenient to construct bosonizations.

Definition 3.3 *[18] The generators of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ are $x_{i,n}^{\pm}$, $h_{i,m}$, h, c $(1 \leq i \leq N, n \in \mathbf{Z}, m \in \mathbf{Z}_{\neq 0})$. Defining relations are*

$$c : \text{central}, \quad [h_i, h_{j,m}] = 0,$$
$$[h_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|} \delta_{m+n,0},$$
$$[h_i, x_j^{\pm}(z)] = \pm A_{i,j} x_j^{\pm}(z),$$
$$[h_{i,m}, x_j^{+}(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x_j^{+}(z),$$
$$[h_{i,m}, x_j^{-}(z)] = -\frac{[A_{i,j}m]_q}{m} z^m x_j^{-}(z),$$
$$(z_1 - q^{\pm A_{i,j}} z_2) x_i^{\pm}(z_1) x_j^{\pm}(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) x_j^{\pm}(z_2) x_i^{\pm}(z_1) \text{ for } |A_{i,j}| \neq 0,$$
$$[x_i^{\pm}(z_1), x_j^{\pm}(z_2)] = 0 \text{ for } |A_{i,j}| = 0,$$
$$[x_i^{+}(z_1), x_j^{-}(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1}) z_1 z_2} \Big(\delta(q^{-c} z_1/z_2) \psi_i^{+}(q^{\frac{c}{2}} z_2) - \delta(q^{c} z_1/z_2) \psi_i^{-}(q^{-\frac{c}{2}} z_2) \Big),$$
$$\Big(x_i^{\pm}(z_1) x_i^{\pm}(z_2) x_j^{\pm}(z) - (q+q^{-1}) x_i^{\pm}(z_1) x_j^{\pm}(z) x_i^{\pm}(z_2) + x_j^{\pm}(z) x_i^{\pm}(z_1) x_i^{\pm}(z_2) \Big)$$
$$+ (z_1 \leftrightarrow z_2) = 0, \qquad \text{for } |A_{i,j}| = 1, \ i \neq N,$$

where we have used $\delta(z) = \sum_{m \in \mathbf{Z}} z^m$. Here we have used the generating function

$$x_j^{\pm}(z) = \sum_{m \in \mathbf{Z}} x_{j,m}^{\pm} z^{-m-1},$$
$$\psi_i^{\pm}(q^{\pm \frac{c}{2}} z) = q^{\pm h_i} e^{\pm (q-q^{-1}) \sum_{m>0} h_{i,\pm m} z^{\mp m}}.$$

The relation between two definitions of $U_q(\widehat{sl}(N|1))$ are given by

$$h_0 = c - (h_1 + \cdots + h_N), \quad e_i = x_{i,0}^{+}, \quad f_i = x_{i,0}^{-} \quad \text{for} \quad 1 \leq i \leq N,$$
$$e_0 = (-1)[x_{N,0}^{-} \cdots, [x_{3,0}^{-}, [x_{2,0}^{-}, x_{1,1}^{-}]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} q^{-h_1 - h_2 - \cdots - h_N},$$
$$f_0 = q^{h_1 + h_2 + \cdots + h_N} [\cdots [[x_{1,-1}^{+}, x_{2,0}^{+}]_q, x_{3,0}^{+}]_q, \cdots x_{N,0}^{+}]_q.$$

For instance we have the coproduct as follows.

$$\Delta(h_{i,m}) = h_{i,m} \otimes q^{\frac{cm}{2}} + q^{\frac{3cm}{2}} \otimes h_{i,m} \quad (m > 0),$$
$$\Delta(h_{i,-m}) = h_{i,-m} \otimes q^{-\frac{3cm}{2}} + q^{-\frac{cm}{2}} \otimes h_{i,-m} \quad (m > 0).$$

3.2. Bosonization

In this section we construct bosonizations of quantum superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$ [2]. We introduce the bosons and the zero-mode operators a_m^j, Q_a^j $(m \in \mathbf{Z}, 1 \le j \le N)$, $b_m^{i,j}, Q_b^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le N+1)$, $c_m^{i,j}, Q_c^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le N)$ which satisfy

$$[a_m^i, a_n^j] = \frac{[(k+N-1)m]_q [A_{i,j} m]_q}{m} \delta_{m+n,0}, \ [a_0^i, Q_a^j] = (k+N-1)A_{i,j},$$
$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \ [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'},$$
$$[c_m^{i,j}, c_n^{i',j'}] = \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \ [c_0^{i,j}, Q_c^{i',j'}] = \delta_{i,i'} \delta_{j,j'},$$
$$[Q_b^{i,j}, Q_b^{i',j'}] = \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad (i,j) \neq (i',j').$$

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings ::. It is convenient to introduce the generating function $b^{i,j}(z), c^{i,j}(z), b_\pm^{i,j}(z), a_\pm^j(z)$ and $\left(\frac{\gamma_1}{\beta_1} \cdots \frac{\gamma_r}{\beta_r} a^i\right)(z|\alpha)$ given by

$$b^{i,j}(z) = -\sum_{m \neq 0} \frac{b_m^{i,j}}{[m]_q} z^{-m} + Q_b^{i,j} + b_0^{i,j} \log z,$$
$$c^{i,j}(z) = -\sum_{m \neq 0} \frac{c_m^{i,j}}{[m]_q} z^{-m} + Q_c^{i,j} + c_0^{i,j} \log z,$$
$$b_\pm^{i,j}(z) = \pm(q - q^{-1}) \sum_{\pm m > 0} b_m^{i,j} z^{-m} \pm b_0^{i,j} \log q,$$
$$a_\pm^j(z) = \pm(q - q^{-1}) \sum_{\pm m > 0} a_m^j z^{-m} \pm a_0^j \log q,$$
$$\left(\frac{\gamma_1}{\beta_1} \cdots \frac{\gamma_r}{\beta_r} \, a^i\right)(z|\alpha) = -\sum_{m \neq 0} \frac{[\gamma_1 m]_q \cdots [\gamma_r m]_q}{[\beta_1 m]_q \cdots [\beta_r m]_q} \frac{a_m^i}{[m]_q} q^{-\alpha|m|} z^{-m}$$
$$+ \frac{\gamma_1 \cdots \gamma_r}{\beta_1 \cdots \beta_r} (Q_a^i + a_0^i \log z).$$

In order to avoid divergence we work on the Fock space defined below. We introduce the vacuum state $|0\rangle \neq 0$ of the boson Fock space by

$$a_m^i |0\rangle = b_m^{i,j} |0\rangle = c_m^{i,j} |0\rangle = 0 \quad (m \ge 0).$$

For $p_a^i \in \mathbf{C}$ $(1 \le i \le N)$, $p_b^{i,j} \in \mathbf{C}$ $(1 \le i < j \le N+1)$, $p_c^{i,j} \in \mathbf{C}$ $(1 \le i < j \le N)$, we set

$$\begin{aligned}|p_a, p_b, p_c\rangle &= e^{\sum_{i,j=1}^N \frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(N-1)(k+N-1)} p_a^i Q_a^j} \\ &\times e^{-\sum_{1\le i<j\le N+1} p_b^{i,j} Q_b^{i,j} + \sum_{1\le i<j\le N} p_c^{i,j} Q_c^{i,j}} |0\rangle.\end{aligned}$$

It satisfies

$$\begin{aligned}&a_0^i |p_a, p_b, p_c\rangle = p_a^i |p_a, p_b, p_c\rangle, \\ &b_0^{i,j} |p_a, p_b, p_c\rangle = p_b^{i,j} |p_a, p_b, p_c\rangle, \ c_0^{i,j} |p_a, p_b, p_c\rangle = p_c^{i,j} |p_a, p_b, p_c\rangle.\end{aligned}$$

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$ on the vector $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z} \ (1 \le i < j \le N) \\ p_b^{i,N+1} \in \mathbf{Z} \ (1 \le i \le N)}} F(p_a, p_b, p_c).$$

We impose the restriction $p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z}$ $(1 \le i < j \le N)$. We construct a bosonization on the space $F(p_a)$.

Theorem 3.4 *[2] A bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$ is given as follows.*

$$c = k \in \mathbf{C},$$

$$h_i = a_0^i + \sum_{l=1}^{i} (b_0^{l,i+1} - b_0^{l,i}) + \sum_{l=i+1}^{N} (b_0^{i,l} - b_0^{i+1,l}) + b_0^{i,N+1} - b_0^{i+1,N+1},$$

$$h_N = a_0^N - \sum_{l=1}^{N-1} (b_0^{l,N} + b_0^{l,N+1}),$$

$$\begin{aligned}h_{i,m} &= q^{-\frac{N-1}{2}|m|} a_m^i + \sum_{l=1}^{i} (q^{-(\frac{k}{2}+l-1)|m|} b_m^{l,i+1} - q^{-(\frac{k}{2}+l)|m|} b_m^{l,i}) \\ &+ \sum_{l=i+1}^{N} (q^{-(\frac{k}{2}+l)|m|} b_m^{i,l} - q^{-(\frac{k}{2}+l-1)|m|} b_m^{i+1,l}) \\ &+ q^{-(\frac{k}{2}+N)|m|} b_m^{i,N+1} - q^{-(\frac{k}{2}+N-1)|m|} b_m^{i+1,N+1},\end{aligned}$$

$$h_{N,m} = q^{-\frac{N-1}{2}|m|} a_m^N - \sum_{l=1}^{N-1} (q^{-(\frac{k}{2}+l)|m|} b_m^{l,N} + q^{-(\frac{k}{2}+l)|m|} b_m^{l,N+1}),$$

$$x_i^+(z) = \frac{1}{(q-q^{-1})z} : \sum_{j=1}^{i} e^{(b+c)^{j,i}(q^{j-1}z) + \sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z))} \times$$

$$
\times \left\{ e^{b_+^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^j z)} - e^{b_-^{j,i+1}(q^{j-1}z)-(b+c)^{j,i+1}(q^{j-2}z)} \right\} :,
$$

$$
x_N^+(z) =: \sum_{j=1}^{N} e^{(b+c)^{j,N}(q^{j-1}z)+b^{j,N+1}(q^{j-1}z)-\sum_{l=1}^{j-1}(b_+^{l,N+1}(q^l z)+b_+^{l,N}(q^l z))} :,
$$

$$
\begin{aligned}
x_i^-(z) = q^{k+N-1} :& e^{a_+^i(q^{\frac{k+N-1}{2}}z)-b^{i,N+1}(q^{k+N-1}z)-b_+^{i+1,N+1}(q^{k+N-1}z)+b^{i+1,N+1}(q^{k+N}z)} : \\
&+ \frac{1}{(q-q^{-1})z} : \left\{ \sum_{j=1}^{i-1} e^{a_-^i(q^{-\frac{k+N-1}{2}}z)+(b+c)^{j,i+1}(q^{-k-j}z)+b_-^{i,n+1}(q^{-k-n}z)} \right. \\
&\times e^{-b_-^{i+1,n+1}(q^{-k-n+1}z)}\, e^{\sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z)-b_-^{l,i}(q^{-k-l}z))} \\
&\times e^{\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))} \\
&\times \left(e^{-b_-^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j+1}z)} - e^{-b_+^{j,i}(q^{-k-j}z)-(b+c)^{j,i}(q^{-k-j-1}z)} \right) \\
&+ e^{a_-^i(q^{-\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{-k-i}z)} \\
&\times e^{\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z)-b_-^{i+1,l}(q^{-k-l+1}z))+b_-^{i,N+1}(q^{-k-N}z)-b_-^{i+1,N+1}(q^{-k-N+1}z)} \\
&- e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,i+1}(q^{k+i}z)} \\
&\times e^{\sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))+b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)} \\
&- \sum_{j=i+1}^{N-1} e^{a_+^i(q^{\frac{k+N-1}{2}}z)+(b+c)^{i,j+1}(q^{k+j}z)} \\
&\times e^{b_+^{i,N+1}(q^{k+N}z)-b_+^{i+1,N+1}(q^{k+N-1}z)+\sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))} \\
&\times \Big(e^{b_+^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j+1}z)} \\
&\left. - e^{b_-^{i+1,j+1}(q^{k+j}z)-(b+c)^{i+1,j+1}(q^{k+j-1}z)} \Big) \right\} : .
\end{aligned}
$$

$$
\begin{aligned}
x_N^-(z) = \frac{1}{(q-q^{-1})z} :& \left\{ \sum_{j=1}^{N-1} e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b_+^{j,N+1}(q^{-k-j}z)-b^{j,N+1}(q^{-k-j-1}z)} \right. \\
&\times e^{-\sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z)+b_-^{l,N+1}(q^{-k-l}z))}\, q^{j-1} \left(e^{-b_+^{j,N}(q^{-k-j}z)} \right) \\
&\times q^{j-1} \left(e^{-(b+c)^{j,N}(q^{-k-j-1}z)} - e^{-b_-^{j,N}(q^{-k-j}z)-(b+c)^{j,N}(q^{-k-j+1}z)} \right) : \\
&+ q^{N-1} : \left(e^{a_+^N(q^{\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{k+N-1}z)}\, e^{a_-^N(q^{-\frac{k+N-1}{2}}z)-b^{N,N+1}(q^{-k-N+1}z)} \right) :.
\end{aligned}
$$

3.3. Replacement from $U_q(sl(N|1))$ to $U_q(\widehat{sl}(N|1))$

In this section we study the relation between $U_q(sl(N|1))$ and $U_q(\widehat{sl}(N|1))$. Let us recall the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$

[1]. We introduce the coordinates $x_{i,j}$, $(1 \le i < j \le N+1)$ by

$$x_{i,j} = \begin{cases} z_{i,j} & (1 \le i < j \le N), \\ \theta_{i,j} & (1 \le i \le N, j = N+1). \end{cases} \tag{3..1}$$

Here $z_{i,j}$ are complex variables and $\theta_{i,N+1}$ are the Grassmann odd variables that satisfy $\theta_{i,N+1}\theta_{i,N+1} = 0$ and $\theta_{i,N+1}\theta_{j,N+1} = -\theta_{j,N+1}\theta_{i,N+1}$, $(i \ne j)$. We introduce the differential operators $\vartheta_{i,j} = x_{i,j}\frac{\partial}{\partial x_{i,j}}$, $(1 \le i < j \le N+1)$.

Theorem 3.5 *[1] We fix parameters $\lambda_i \in \mathbf{C}$ $(1 \le i \le N)$. The Heisenberg realization of $U_q(sl(N|1))$ is given as follows.*

$$h_i = \sum_{j=1}^{i-1}(\nu_i\vartheta_{j,i} - \nu_{i+1}\vartheta_{j,i+1}) + \lambda_i - (\nu_i + \nu_{i+1})\vartheta_{i,i+1} + \sum_{j=i+1}^{N}(\nu_{i+1}\vartheta_{i+1,j+1} - \nu_i\vartheta_{i,j+1}),$$

$$e_i = \sum_{j=1}^{i}\frac{x_{j,i}}{x_{j,i+1}}[\vartheta_{j,i+1}]_q\, q^{\sum_{l=1}^{j-1}(\nu_i\vartheta_{l,i} - \nu_{i+1}\vartheta_{l,i+1})},$$

$$f_i = \sum_{j=1}^{i-1}\nu_i\frac{x_{j,i+1}}{x_{j,i}}[\vartheta_{j,i}]_q q^{\sum_{l=j+1}^{i-1}(\nu_{i+1}\vartheta_{l,i+1} - \nu_i\vartheta_{l,i}) - \lambda_i + (\nu_i + \nu_{i+1})\vartheta_{i,i+1} + \sum_{l=i+2}^{N+1}(\nu_i\vartheta_{i,l} - \nu_{i+1}\vartheta_{i+1,l})}$$
$$+ x_{i,i+1}\left[\lambda_i - \nu_i\vartheta_{i,i+1} - \sum_{l=i+2}^{N+1}(\nu_i\vartheta_{i,l} - \nu_{i+1}\vartheta_{i+1,l})\right]_q$$
$$- \sum_{j=i+1}^{N}\nu_{i+1}\frac{x_{i,j+1}}{x_{i+1,j+1}}[\vartheta_{i+1,j+1}]_q q^{\lambda_i + \sum_{l=j+1}^{N+1}(\nu_{i+1}\vartheta_{i+1,l} - \nu_i\vartheta_{i,l})}.$$

Here we read $x_{i,i} = 1$ and, for Grassmann odd variables $x_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for the derivative $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$.

We study how to recover the bosonization of the affine superalgebra $U_q(\widehat{sl}(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

$$\begin{aligned}
\vartheta_{i,j} &\to -b_\pm^{i,j}(z)/\log q \quad (1 \le i < j \le N+1), \\
[\vartheta_{i,j}]_q &\to \begin{cases} \dfrac{e^{\pm b_+^{i,j}(z)} - e^{\pm b_-^{i,j}(z)}}{(q-q^{-1})z} & (j \ne N+1), \\ 1 & (j = N+1). \end{cases} \\
x_{i,j} &\to \begin{cases} :e^{(b+c)^{i,j}(z)}: & (j \ne N+1), \\ :e^{-b^{i,j}(z)}: \text{ or } :e^{-b_\pm^{i,j}(q^{\pm 1}z) - b^{i,j}(z)}: & (j = N+1). \end{cases} \\
\lambda_i &\to a_\pm^i(z)/\log q \quad (1 \le i \le N), \\
[\lambda_i]_q &\to \frac{e^{\pm a_+^i(z)} - e^{\pm a_-^i(z)}}{(q-q^{-1})z} \quad (1 \le i \le N).
\end{aligned}$$

From the above replacement, the element h_i of the Heisenberg realization is replaced as following.

$$q^{h_i} \to \begin{cases} e^{a_\pm^i(z)+\sum_{l=1}^{i}(b_\pm^{l,i+1}(z)-b_\pm^{l,i}(z))+\sum_{l=i+1}^{N}(b_\pm^{i,l}(z)-b_\pm^{i+1,l}(z))}, & (1\le i\le N-1),\\ e^{a_\pm^N(z)-\sum_{l=1}^{N-1}(b_\pm^{l,N}(z)+b_\pm^{l,N+1}(z))}, & (i=N). \end{cases}$$

We impose q-shift to variable z of the operators $a_\pm^i(z)$, $b_\pm^{i,j}(z)$. For instance, we have to replace $a_\pm^i(z) \to a_\pm^i(q^{\pm\frac{c+N-1}{2}}z)$. Bridging the gap by the q-shift, we have the bosonizations $\psi_i^\pm(q^{\pm\frac{c}{2}}z) \in U_q(\widehat{sl}(N|1))$ from $q^{h_i} \in U_q(sl(N|1))$.

$$\psi_i^\pm(q^{\pm\frac{c}{2}}z) = e^{a_\pm^i(q^{\pm\frac{c+N-1}{2}}z)+\sum_{l=1}^{i}(b_\pm^{l,i+1}(q^{\pm(l+c-1)}z)-b_\pm^{l,i}(q^{\pm(l+c)}z))}$$
$$\times e^{\sum_{l=i+1}{}^{N}(b_\pm^{i,l}(q^{\pm(c+l)}z)-b_\pm^{i-1,l}(q^{\pm(c+l-1)}z))+b_\pm^{i,N+1}(q^{\pm(c+N)}z)\ -b_\pm^{i+1,N+1}(q^{\pm(c+N-1)}z)},$$
$$\psi_N^\pm(q^{\pm\frac{c}{2}}z) = e^{a_\pm^N(q^{\pm\frac{c+N-1}{2}}z)-\sum_{l=1}^{N-1}(b_\pm^{l,N}(q^{\pm(c+l)}z)+b_\pm^{l,N+1}(q^{\pm(c+l)}z))}.$$

In this replacement, one element q^{h_i} goes to two elements $\psi_i^\pm(q^{\pm\frac{c}{2}}z)$. Hence this replacement is not a map. Replacements from e_i, f_i to $x_i^\pm(z)$ are given by similar way, however they are more complicated. See details in [2].

3.4. Wakimoto Realization

In this section we give the Wakimoto realization $\mathcal{F}(p_a)$ whose character coincides with those of the Verma module [14]. We introduce the operators $\xi_m^{i,j}$ and $\eta_m^{i,j}$ $(1\le i<j\le N, m\in\mathbf{Z})$ by

$$\eta^{i,j}(z)=\sum_{m\in\mathbf{Z}}\eta_m^{i,j}z^{-m-1} =: e^{c^{i,j}(z)} :,\quad \xi^{i,j}(z)=\sum_{m\in\mathbf{Z}}\xi_m^{i,j}z^{-m} =: e^{-c^{i,j}(z)} : .$$

The Fourier components $\eta_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^m\eta^{i,j}(z)$, $\xi_m^{i,j}=\oint\frac{dz}{2\pi\sqrt{-1}}z^{m-1}\xi^{i,j}(z)$ $(m\in\mathbf{Z})$ are well defined on the space $F(p_a)$. We focus our attention on the operators $\eta_0^{i,j},\xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2=0$, $(\xi_0^{i,j})^2=0$. They satisfy

$$\mathrm{Im}(\eta_0^{i,j})=\mathrm{Ker}(\eta_0^{i,j}),\ \mathrm{Im}(\xi_0^{i,j})=\mathrm{Ker}(\xi_0^{i,j}),\ \eta_0^{i,j}\xi_0^{i,j}+\xi_0^{i,j}\eta_0^{i,j}=1.$$

We have a direct sum decomposition.

$$F(p_a)=\eta_0^{i,j}\xi_0^{i,j}F(p_a)\oplus\xi_0^{i,j}\eta_0^{i,j}F(p_a),$$
$$\mathrm{Ker}(\eta_0^{i,j})=\eta_0^{i,j}\xi_0^{i,j}F(p_a),\ \mathrm{Coker}(\eta_0^{i,j})=\xi_0^{i,j}\eta_0^{i,j}F(p_a)=F(p_a)/(\eta_0^{i,j}\xi_0^{i,j})F(p_a).$$

We set the operator η_0,ξ_0 by

$$\eta_0=\prod_{1\le i<j\le N}\eta_0^{i,j},\quad \xi_0=\prod_{1\le i<j\le N}\xi_0^{i,j}.$$

Definition 3.6 *[14] We introduce the subspace $\mathcal{F}(p_a)$ by*

$$\mathcal{F}(p_a)=\eta_0\xi_0F(p_a).$$

We call $\mathcal{F}(p_a)$ the Wakimoto realization.

4. Screening and Vertex Operator

In this section we give the screening that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$. We propose the vertex operators and the correlation functions.

4.1. Screening

In this section we give the screening $\mathcal{Q}_i$ $(1 \leq i \leq N)$ that commutes with the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \neq -N+1$ [15]. The Jackson integral with parameter $p \in \mathbf{C}$ $(|p| < 1)$ and $s \in \mathbf{C}^*$ is defined by

$$\int_0^{s\infty} f(z) d_p z = s(1-p) \sum_{m \in \mathbf{Z}} f(sp^m) p^m.$$

In order to avoid divergence we work in the Fock space.

Theorem 4.1 *[15] The screening $\mathcal{Q}_i$ commutes with the quantum superalgebra.*

$$[\mathcal{Q}_i, U_q(\widehat{sl}(N|1))] = 0 \quad (1 \leq i \leq N).$$

We have introduced the screening operators $\mathcal{Q}_i$ $(1 \leq i \leq N)$ as follows.

$$\mathcal{Q}_i = \int_0^{s\infty} : e^{-\left(\frac{1}{k+N-1} a^i\right)\left(z \middle| \frac{k+N-1}{2}\right)} \widetilde{S}_i(z) : d_p z, \quad (p = q^{2(k+N-1)}).$$

Here we have set the bosonic operators $\widetilde{S}_i(z)$ $(1 \leq i \leq N)$ by

$$\begin{aligned}
\widetilde{S}_i(z) = & \frac{1}{(q-q^{-1})z} \sum_{j=i+1}^{N} : \left(e^{-b_-^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j}z)} \right. \\
& \left. - e^{-b_+^{i,j}(q^{N-1-j}z) - (b+c)^{i,j}(q^{N-j-2}z)} \right) e^{(b+c)^{i+1,j}(q^{N-1-j}z)} \\
& \times e^{\sum_{l=j+1}^{N} (b_-^{i+1,l}(q^{N-l}z) - b_-^{i,l}(q^{N-l-1}z)) + b_-^{i+1,N+1}(z) - b_-^{i,N+1}(q^{-1}z)} : \\
& + q : e^{b^{i,N+1}(z) + b_+^{i+1,N+1}(z) - b^{i+1,N+1}(qz)} : \qquad (1 \leq i \leq N-1), \\
\widetilde{S}_N(z) = & -q^{-1} : e^{b^{N,N+1}(z)} : .
\end{aligned}$$

4.2. Vertex Operator

In this section we introduce the vertex operators $\Phi(z)$, $\Phi^*(z)$ [15]. Let $\mathcal{F}$ and $\mathcal{F}'$ be $U_q(\widehat{sl}(N|1))$ representation for an arbitrary level $k \neq -N+1$. Let

V_α and V_α^{*S} be 2^N-dimensional typical representation with a parameters α [21]. Let $\{v_j\}_{j=1}^{2^N}$ be the basis of V_α. Let $\{v_j^*\}_{j=1}^{2^N}$ be the dual basis of V_α^{*S}, satisfying $(v_i|v_j^*)=\delta_{i,j}$. Let $V_{\alpha,z}$ and $V_{\alpha,z}^{*S}$ be the evaluation module and its dual of the typical representation. For instance, the 8-dimensional representation $V_{\alpha,z}$ of $U_q(\widehat{sl}(3|1))$ is given by

$$\begin{aligned}
h_1 &= E_{3,3}-E_{4,4}+E_{5,5}-E_{6,6},\\
h_2 &= E_{2,2}-E_{3,3}+E_{6,6}-E_{7,7},\\
h_3 &= \alpha(E_{1,1}+E_{2,2})+(\alpha+1)(E_{3,3}+E_{4,4}+E_{5,5}+E_{6,6})+(\alpha+2)(E_{7,7}+E_{8,8}),\\
e_1 &= E_{3,4}+E_{5,6},\\
e_2 &= E_{2,3}+E_{6,7},\\
e_3 &= \sqrt{[\alpha]_q}E_{1,2}-\sqrt{[\alpha+1]_q}(E_{3,5}+E_{4,6})+\sqrt{[\alpha+2]_q}E_{7,8},\\
f_1 &= E_{4,3}+E_{6,5},\\
f_2 &= E_{3,2}+E_{7,6},\\
f_3 &= \sqrt{[\alpha]_q}E_{2,1}-\sqrt{[\alpha+1]_q}(E_{5,3}+E_{6,4})+\sqrt{[\alpha+2]_q}E_{8,7},\\
h_0 &= -\alpha(E_{1,1}+E_{4,4})-(\alpha+1)(E_{2,2}+E_{3,3}+E_{6,6}+E_{7,7})-(\alpha+2)(E_{5,5}+E_{8,8}),\\
e_0 &= -z(\sqrt{[\alpha]_q}E_{4,1}-\sqrt{[\alpha+1]_q}(E_{6,2}+E_{7,3})+\sqrt{[\alpha+2]_q}E_{8,5}),\\
f_0 &= z^{-1}(\sqrt{[\alpha]_q}E_{1,4}-\sqrt{[\alpha+1]_q}(E_{2,6}+E_{3,7})+\sqrt{[\alpha+2]_q}E_{5,8}).
\end{aligned}$$

Consider the following intertwiners of $U_q(\widehat{sl}(N|1))$-representation [20].

$$\Phi(z):\mathcal{F}\longrightarrow\mathcal{F}'\otimes V_{\alpha,z},\quad \Phi^*(z):\mathcal{F}\longrightarrow\mathcal{F}'\otimes V_{\alpha,z}^{*S}.$$

They are intertwiners in the sense that for any $x\in U_q(\widehat{sl}(N|1))$,

$$\Phi(z)\cdot x=\Delta(x)\cdot\Phi(z),\quad \Phi^*(z)\cdot x=\Delta(x)\cdot\Phi^*(z).$$

We expand the intertwining operators.

$$\Phi(z)=\sum_{j=1}^{2^N}\Phi_j(z)\otimes v_j,\quad \Phi^*(z)=\sum_{j=1}^{2^N}\Phi_j^*(z)\otimes v_j^*.$$

We set the $\mathbf{Z}_2$-grading of the intertwiner be $|\Phi(z)|=|\Phi^*(z)|=0$. For $l_a=(l_a^1,l_a^2,\cdots,l_a^N)\in\mathbf{C}^N$ and $\beta\in\mathbf{C}$, we set the bosonic operator $\phi^{l_a}(z|\beta)$ by

$$\phi^{l_a}(z|\beta)=:e^{\sum_{i,j=1}^N\left(\frac{l_a^i}{k+N-1}\frac{\mathrm{Min}(i,j)}{N-1}\frac{N-1-\mathrm{Max}(i,j)}{1}a^j\right)(z|\beta)}:.$$

In order to balance thegbackground chargeh of the vertex operators, we introduce the product of the screenings $\mathcal{Q}^{(t)}$ for $t=(t_1,t_2,\cdots,t_N)\in\mathbf{N}^N$.

$$\mathcal{Q}^{(t)}=\mathcal{Q}_1^{t_1}\mathcal{Q}_2^{t_2}\cdots\mathcal{Q}_N^{t_N}.$$

The screening operator $\mathcal{Q}^{(t)}$ give rise to the map,

$$\mathcal{Q}^{(t)}:\mathcal{F}(p_a)\to\mathcal{F}(p_a+\hat{t}).$$

Here $\hat{t}=(\hat{t}_1,\hat{t}_2,\cdots,\hat{t}_N)$ where $\hat{t}_i=\sum_{j=1}^N A_{i,j}t_j$.

Theorem 4.2 *[15] For $k=\alpha\neq 0,-1,-2,\cdots,-N+1$, bosonizations of the special components of the vertex operators $\Phi^{(t)}(z)$ and $\Phi^{*(t)}(z)$ are given by*

$$\begin{aligned}\Phi_{2^N}^{(t)}(z) &= \mathcal{Q}^{(t)}\phi^{\hat{l}}\left(q^{k+N-1}z\left|-\frac{k+N-1}{2}\right.\right),\\ \Phi_1^{*(t)}(z) &= \mathcal{Q}^{(t)}\phi^{\hat{l}^*}\left(q^{k}z\left|-\frac{k+N-1}{2}\right.\right),\end{aligned}$$

where we have used $\hat{l}=-(0,\cdots,0,\alpha+N-1)$, $\hat{l}^*=(0,\cdots,0,\alpha)$ and $t=(t_1,t_2,\cdots,t_N)\in\mathbf{N}^N$. The other components $\Phi_j^{(t)}(z)$ and $\Phi_j^{*(t)}(z)$ $(1\le j\le 2^N)$ are determined by the intertwining property and are represented by multiple contour integrals of Drinfeld currents and the special components $\Phi_{2^N}^{(t)}(z)$ and $\Phi_1^{*(t)}(z)$. We have checked this theorem for $N=2,3,4$.

Here we give additional explanation on the above theorem. The explicit formulae of the intertwining properties $\Phi^{(t)}(z)\cdot x=\Delta(x)\cdot\Phi^{(t)}(z)$ for $U_q(\widehat{sl}(3|1))$ are summarized as follows. We have set the $\mathbf{Z}_2$-grading of V_α as follows : $|v_1|=|v_5|=|v_6|=|v_7|=0$, and $|v_2|=|v_3|=|v_4|=|v_8|=1$.

$$\begin{aligned}&\Phi_3^{(t)}(z)=[\Phi_4^{(t)}(z),f_1]_q,\ \Phi_5^{(t)}(z)=[\Phi_6^{(t)}(z),f_1]_q,\\ &\Phi_2^{(t)}(z)=[\Phi_3^{(t)}(z),f_2]_q,\ \Phi_6^{(t)}(z)=[\Phi_7^{(t)}(z),f_2]_q,\\ &\Phi_1^{(t)}(z)=\frac{1}{\sqrt{[\alpha]_q}}[\Phi_2^{(t)}(z),f_3]_{q^{-\alpha}},\ \Phi_3^{(t)}(z)=\frac{-1}{\sqrt{[\alpha+1]_q}}[\Phi_5^{(t)}(z),f_3]_{q^{-\alpha-1}},\\ &\Phi_4^{(t)}(z)=\frac{-1}{\sqrt{[\alpha+1]_q}}[\Phi_6^{(t)}(z),f_3]_{q^{-\alpha-1}},\ \Phi_7^{(t)}(z)=\frac{1}{\sqrt{[\alpha+2]_q}}[\Phi_8^{(t)}(z),f_3]_{q^{-\alpha-2}}.\end{aligned}$$

The elements f_j are written by contour integral of the Drinfeld current $f_j=\oint\frac{dw}{2\pi\sqrt{-1}}x_j^-(w)$. Hence the components $\Phi_j^{(t)}$ $(1\le j\le 8)$ are represented by multiple contour integrals of Drinfeld currents $x_j^-(w)$ $(1\le j\le 3)$ and the special component $\Phi_8^{(t)}(z)$.

4.3. Correlation Function

In this section we study the correlation function as an application of the vertex operators . We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Wakimoto module of $U_q(\widehat{sl}(N|1))$. We propose the q-Virasoro operator L_0 for $k = \alpha \neq -N+1$ as follows.

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_{i,j=1}^{N}\sum_{m\in\mathbf{Z}} : a_{-m}^{i} \frac{m^2[\mathrm{Min}(i,j)m]_q[(N-1-\mathrm{Max}(i,j))m]_q}{[m]_q[(k+N-1)m]_q[(N-1)m]_q[m]_q} a_m^j : \\ &+ \sum_{i,j=1}^{N} \frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(k+N-1)(N-1)} a_0^j \\ &- \frac{1}{2}\sum_{1\le i<j\le N}\sum_{m\in\mathbf{Z}} : b_{-m}^{i,j}\frac{m^2}{[m]_q^2} b_m^{i,j} : + \frac{1}{2}\sum_{1\le i<j\le N}\sum_{m\in\mathbf{Z}} : c_{-m}^{i,j}\frac{m^2}{[m]_q^2} c_m^{i,j} : \\ &+ \frac{1}{2}\sum_{1\le i\le N}\sum_{m\in\mathbf{Z}} : b_{-m}^{i,N+1}\frac{m^2}{[m]_q^2} b_m^{i,N+1} : + \frac{1}{2}\sum_{1\le i\le N} b_0^{i,N+1}. \end{aligned}$$

The L_0 eigenvalue of $|l_a, 0, 0\rangle$ is $\frac{1}{2(k+N-1)}(\bar{\lambda}|\bar{\lambda}+2\bar{\rho})$, where $\bar{\rho} = \sum_{i=1}^{N} \bar{\Lambda}_i$ and $\bar{\lambda} = \sum_{i=1}^{N} l_a^i \bar{\Lambda}_i$.

Theorem 4.3 *[15] For $k = \alpha \neq 0, -1, -2, \cdots, -N+1$, the correlation function of the vertex operators,*

$$\mathrm{Tr}_{\mathcal{F}(l_a)}\left(q^{L_0}\Phi_{i_1}^{*(y_{(1)})}(w_1)\cdots\Phi_{i_m}^{*(y_{(m)})}(w_m)\Phi_{j_1}^{(x_{(1)})}(z_1)\cdots\Phi_{j_n}^{(x_{(n)})}(z_n)\right) \neq 0,$$

if and only if $x_{(s)} = (x_{(s),1}, x_{(s),2}, \cdots, x_{(s),N}) \in \mathbf{N}^N$ $(1 \le s \le n)$ and $y_{(s)} = (y_{(s),1}, y_{(s),2}, \cdots, y_{(s),N}) \in \mathbf{N}^N$ $(1 \le s \le m)$ satisfy the following condition.

$$\sum_{s=1}^{n} x_{(s),i} + \sum_{s=1}^{m} y_{(s),i} = \frac{(n-m)i}{N-1}\alpha + n\cdot i \qquad (1 \le i \le N).$$

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