Equations of Motion in Nonlocal Modified Gravity

Jelena Grujić*
Teachers Training Faculty, University of Belgrade
Kraljice Natalije 43, Belgrade, SERBIA

Abstract

We consider nonlocal modified gravity without matter, where nonlocality is of the form $\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)$. Equations of motion are usually very complex and their derivation is not available in the research papers. In this paper we give the corresponding derivation in detail.

Key words: Modified gravity, nonlocal gravity, variational principle, equations of motion.

1. Introduction

Einstein's theory of gravity, which is General Relativity, is given by Einstein-Hilbert action which Lagrangian contains Ricci scalar in the linear form. The first attempts to modify Einstein's theory of gravity started soon after its appearance and they were mainly inspired by investigation of possible mathematical generalizations. The discovery of accelerating expansion of the Universe in 1998 has not so far generally accepted theoretical explanation and it has produced an intensive activity in general relativity modification during the last decade.

In this paper, we consider nonlocal modification of gravity (for a review, see [1, 2, 3, 4]). Under nonlocal modification of gravity we understand replacement of the scalar curvature R in the Einstein-Hilbert action by a suitable function $F(R, \Box)$, where $\Box = \nabla_\mu \nabla^\mu$ is d'Alembert operator and ∇_μ denotes the covariant derivative. Here, nonlocality means that Lagrangian contains an infinite number of space-time derivatives, i.e. derivatives up to an infinite order in the form of d'Alembert operator $\Box$ which is argument of an analytic function.

We consider a class of nonlocal gravity models without matter, given by the next action

$$S = \int d^4x \sqrt{-g}\Big(\frac{R - 2\Lambda}{16\pi G} + C\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\Big), \tag{1}$$

* e-mail address: jelenagg@gmail.com

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, $\mathcal{H}$ and $\mathcal{G}$ are differentiable functions of the scalar curvature R, Λ is cosmological constant and C is a constant. The corresponding Einstein equations of motion are rather complex and their derivation is not available in the research papers. In this paper we will present their derivation. In order to obtain equations of motion for $g_{\mu\nu}$ we must find the variation of the action (1) with respect to metric $g^{\mu\nu}$.

2. Some preliminaries

At the beginning we would like to prove the following identities:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}, \tag{2}$$

$$\delta\sqrt{-g} = -\frac{1}{2} g_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu}, \tag{3}$$

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}, \tag{4}$$

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma} = -g^{\mu\alpha} \Gamma^\nu_{\sigma\alpha} - g^{\nu\alpha} \Gamma^\mu_{\sigma\alpha}, \tag{5}$$

$$\Gamma^\mu_{\mu\nu} = \frac{\partial}{\partial x^\nu} \ln \sqrt{-g}, \tag{6}$$

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu, \tag{7}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$.

The determinant g can be expressed in the following way:

$$g = g_{\mu 0} G^{(\mu,0)} + g_{\mu 1} G^{(\mu,1)} + ... + g_{\mu n-1} G^{(\mu,n-1)}, \tag{8}$$

where $G^{(\mu,\nu)}$ is the corresponding algebraic cofactor and n is space-time dimension. If we replace elements of the μ row with the elements of the ν row then the determinant g becomes equal zero, i.e.

$$0 = g_{\nu 0} G^{(\mu,0)} + g_{\nu 1} G^{(\mu,1)} + ... + g_{\nu n-1} G^{(\mu,n-1)}. \tag{9}$$

From this expression we have

$$g_{\mu\nu} G^{(\alpha,\nu)} = g \delta^\alpha_\mu, \tag{10}$$

or

$$g_{\mu\nu} \frac{G^{(\alpha,\nu)}}{g} = \delta^\alpha_\mu, \tag{11}$$

where δ^α_μ is Kronecker's delta symbol.

Metric tensor $g^{\alpha\nu}$ is defined by

$$g^{\alpha\nu} = \frac{G^{(\alpha,\nu)}}{g}. \tag{12}$$

From (10) (here we set $\alpha = \mu$) we get

$$\delta g = G^{(\mu,\nu)} \delta g_{\mu\nu}. \tag{13}$$

From the last equation and from (12) we obtain

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{14}$$

From $g_{\mu\nu} g^{\mu\nu} = n$ we have $g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$ which completes the proof for (2).
We have

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g. \tag{15}$$

Finally, using equation (2) we obtain (3).

We know that by definition

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{16}$$

where $R_{\mu\nu} = R^{\eta}_{\mu\eta\nu}$ is the Ricci tensor.
The Riemann tensor $R^{\mu}_{\nu\gamma\eta}$ is given by

$$R^{\mu}_{\nu\gamma\eta} = \frac{\partial \Gamma^{\mu}_{\nu\eta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\mu}_{\gamma\nu}}{\partial x^{\eta}} + \Gamma^{\mu}_{\gamma\sigma}\Gamma^{\sigma}_{\nu\eta} - \Gamma^{\mu}_{\sigma\eta}\Gamma^{\sigma}_{\gamma\nu}, \tag{17}$$

where $\Gamma^{\mu}_{\nu\gamma}$ is the Christoffel symbol

$$\Gamma^{\mu}_{\nu\gamma} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\sigma\gamma}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial x^{\sigma}}\right). \tag{18}$$

It can be shown that

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\beta\nu} \delta g^{\alpha\beta}, \tag{19}$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\beta\nu} \delta g_{\alpha\beta}, \tag{20}$$

$$\delta R^{\mu}_{\nu\gamma\eta} = \nabla_{\gamma}\delta\Gamma^{\mu}_{\eta\nu} - \nabla_{\eta}\delta\Gamma^{\mu}_{\gamma\nu}, \tag{21}$$

$$\delta R_{\mu\nu} = \nabla_{\gamma}\delta\Gamma^{\gamma}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\gamma}_{\gamma\mu}, \tag{22}$$

$$\delta\Gamma^{\sigma}_{\nu\mu} = \frac{1}{2}\delta g^{\sigma\gamma}\left(\frac{\partial g_{\gamma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\nu\gamma}}{\partial x^{\mu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\gamma}}\right) + \frac{1}{2} g^{\sigma\gamma}\left(\frac{\partial \delta g_{\gamma\mu}}{\partial x^{\nu}} + \frac{\partial \delta g_{\nu\gamma}}{\partial x^{\mu}} - \frac{\partial \delta g_{\nu\mu}}{\partial x^{\gamma}}\right). \tag{23}$$

From equation (16) we have that the variation of the Ricci scalar is

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{24}$$

Using equation (22) we get

$$\begin{aligned}\delta R &= \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}(\nabla_\gamma \delta\Gamma^\gamma_{\nu\mu} - \nabla_\nu \delta\Gamma^\gamma_{\gamma\mu}) \\ &= \delta g^{\mu\nu} R_{\mu\nu} + \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\gamma_{\mu\gamma}).\end{aligned} \tag{25}$$

We have used $\nabla_\gamma g_{\mu\nu} = 0$ (the metric compatibility) and relabeled some dummy indices.

Now we need to compute the term $g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\gamma_{\mu\gamma}$. We have

$$\nabla_\gamma \delta g_{\mu\nu} = \frac{\partial \delta g_{\mu\nu}}{\partial x^\gamma} - \Gamma^\sigma_{\gamma\mu}\delta g_{\sigma\nu} - \Gamma^\sigma_{\gamma\nu}\delta g_{\mu\sigma}. \tag{26}$$

Using the last equation, equation (23) and the symmetry in the Christoffel symbol $\Gamma^\mu_{\nu\gamma} = \Gamma^\mu_{\gamma\nu}$ we obtain

$$\begin{aligned}\delta\Gamma^\sigma_{\nu\mu} &= \frac{1}{2}\delta g^{\sigma\gamma}\Big(\frac{\partial g_{\gamma\mu}}{\partial x^\nu} + \frac{\partial g_{\gamma\nu}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\gamma}\Big) + \frac{1}{2} g^{\sigma\gamma}\Big(\nabla_\nu \delta g_{\gamma\mu} + \nabla_\mu \delta g_{\gamma\nu} \\ &\quad - \nabla_\gamma \delta g_{\nu\mu} + \Gamma^\lambda_{\nu\mu}\delta g_{\gamma\lambda} + \Gamma^\lambda_{\mu\nu}\delta g_{\lambda\gamma}\Big) \\ &= \frac{1}{2}\delta g^{\sigma\gamma}\Big(\frac{\partial g_{\gamma\mu}}{\partial x^\nu} + \frac{\partial g_{\gamma\nu}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\gamma}\Big) + g^{\sigma\gamma}\Gamma^\lambda_{\nu\mu}\delta g_{\gamma\lambda} + \frac{1}{2} g^{\sigma\gamma}\Big(\nabla_\nu \delta g_{\gamma\mu} \\ &\quad + \nabla_\mu \delta g_{\gamma\nu} - \nabla_\gamma \delta g_{\nu\mu}\Big).\end{aligned} \tag{27}$$

Now using equation (19) in the second term we get

$$\begin{aligned}\delta\Gamma^\sigma_{\nu\mu} &= \frac{1}{2}\delta g^{\sigma\gamma}\Big(\frac{\partial g_{\gamma\mu}}{\partial x^\nu} + \frac{\partial g_{\gamma\nu}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\gamma}\Big) - \delta g^{\alpha\beta} g^{\sigma\gamma} g_{\gamma\alpha} g_{\lambda\beta}\Gamma^\lambda_{\nu\mu} \\ &\quad + \frac{1}{2} g^{\sigma\gamma}\Big(\nabla_\nu \delta g_{\gamma\mu} + \nabla_\mu \delta g_{\gamma\nu} - \nabla_\gamma \delta g_{\nu\mu}\Big) \\ &= \delta g^{\sigma\beta} g_{\lambda\beta}\Gamma^\lambda_{\nu\mu} - \delta g^{\alpha\beta}\delta^\sigma_\alpha g_{\lambda\beta}\Gamma^\lambda_{\nu\mu} + \frac{1}{2} g^{\sigma\gamma}\Big(\nabla_\nu \delta g_{\gamma\mu} + \nabla_\mu \delta g_{\gamma\nu} \\ &\quad - \nabla_\gamma \delta g_{\nu\mu}\Big) \\ &= \delta g^{\sigma\beta} g_{\lambda\beta}\Gamma^\lambda_{\nu\mu} - \delta g^{\sigma\beta} g_{\lambda\beta}\Gamma^\lambda_{\nu\mu} + \frac{1}{2} g^{\sigma\gamma}\Big(\nabla_\nu \delta g_{\gamma\mu} + \nabla_\mu \delta g_{\gamma\nu} \\ &\quad - \nabla_\gamma \delta g_{\nu\mu}\Big).\end{aligned} \tag{28}$$

Then we have

$$\delta\Gamma^\sigma_{\nu\mu} = \frac{1}{2} g^{\sigma\gamma}(\nabla_\nu \delta g_{\mu\gamma} + \nabla_\mu \delta g_{\nu\gamma} - \nabla_\gamma \delta g_{\nu\mu}). \tag{29}$$

Similarly,

$$\delta\Gamma^\gamma_{\mu\gamma} = \frac{1}{2} g^{\sigma\gamma}\nabla_\mu \delta g_{\sigma\gamma}. \tag{30}$$

In order to express the previous result as function of the variations $\delta g^{\mu\nu}$ we again use (19) and obtain

$$\begin{aligned}\delta\Gamma^{\sigma}_{\nu\mu} &= \frac{1}{2}g^{\sigma\gamma}\Big(\nabla_\nu(-g_{\mu\alpha}g_{\gamma\beta}\delta g^{\alpha\beta}) + \nabla_\mu(-g_{\nu\alpha}g_{\gamma\beta}\delta g^{\alpha\beta}) - \nabla_\gamma(-g_{\nu\alpha}g_{\mu\beta}\delta g^{\alpha\beta})\Big)\\ &= -\frac{1}{2}g^{\sigma\gamma}(g_{\mu\alpha}g_{\gamma\beta}\nabla_\nu\delta g^{\alpha\beta} + g_{\nu\alpha}g_{\gamma\beta}\nabla_\mu\delta g^{\alpha\beta} - g_{\nu\alpha}g_{\mu\beta}\nabla_\gamma\delta g^{\alpha\beta})\\ &= -\frac{1}{2}(\delta^{\sigma}_{\beta}g_{\mu\alpha}\nabla_\nu\delta g^{\alpha\beta} + \delta^{\sigma}_{\beta}g_{\nu\alpha}\nabla_\mu\delta g^{\alpha\beta} - g_{\nu\alpha}g_{\mu\beta}g^{\gamma\sigma}\nabla_\gamma\delta g^{\alpha\beta})\\ &= -\frac{1}{2}(g_{\mu\gamma}\nabla_\nu\delta g^{\sigma\gamma} + g_{\nu\gamma}\nabla_\mu\delta g^{\sigma\gamma} - g_{\nu\alpha}g_{\mu\beta}\nabla^{\sigma}\delta g^{\alpha\beta}), \qquad (31)\end{aligned}$$

where $\nabla^\sigma = g^{\sigma\gamma}\nabla_\gamma$.
Similarly,

$$\delta\Gamma^{\gamma}_{\mu\gamma} = -\frac{1}{2}g_{\alpha\beta}\nabla_\mu\delta g^{\alpha\beta}. \qquad (32)$$

Finally, we obtain

$$\begin{aligned}&g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\gamma}_{\mu\gamma}\\ &\quad= -\frac{1}{2}\Big(g^{\mu\nu}g_{\mu\gamma}\nabla_\nu\delta g^{\sigma\gamma} + g^{\mu\nu}g_{\nu\gamma}\nabla_\mu\delta g^{\sigma\gamma} - g^{\mu\nu}g_{\nu\alpha}g_{\mu\beta}\nabla^{\sigma}\delta g^{\alpha\beta}\\ &\quad- g^{\mu\sigma}g_{\alpha\beta}\nabla_\mu\delta g^{\alpha\beta}\Big)\\ &\quad= -\frac{1}{2}\Big(\delta^{\nu}_{\gamma}\nabla_\nu\delta g^{\sigma\gamma} + \delta^{\mu}_{\gamma}\nabla_\mu\delta g^{\sigma\gamma} - \delta^{\mu}_{\alpha}g_{\mu\beta}\nabla^{\sigma}\delta g^{\alpha\beta} - g^{\mu\sigma}g_{\alpha\beta}\nabla_\mu\delta g^{\alpha\beta}\Big)\\ &\quad= -\frac{1}{2}\Big(\nabla_\gamma\delta g^{\sigma\gamma} + \nabla_\gamma\delta g^{\sigma\gamma} - g_{\alpha\beta}\nabla^{\sigma}\delta g^{\alpha\beta} - g_{\alpha\beta}\nabla^{\sigma}\delta g^{\alpha\beta}\Big)\\ &\quad= -\frac{1}{2}\Big(2\nabla_\gamma\delta g^{\sigma\gamma} - 2g_{\alpha\beta}\nabla^{\sigma}\delta g^{\alpha\beta}\Big). \qquad (33)\end{aligned}$$

Then we have

$$g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\gamma}_{\mu\gamma} = g_{\alpha\beta}\nabla^{\sigma}\delta g^{\alpha\beta} - \nabla_\gamma\delta g^{\sigma\gamma}. \qquad (34)$$

Substituting this in (25) we obtain the variation of the scalar curvature

$$\begin{aligned}\delta R &= \delta g^{\mu\nu}R_{\mu\nu} + g_{\alpha\beta}\nabla_\sigma\nabla^{\sigma}\delta g^{\alpha\beta} - \nabla_\sigma\nabla_\gamma\delta g^{\sigma\gamma}\\ &= \delta g^{\mu\nu}R_{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \qquad (35)\end{aligned}$$

Now we want to prove the equation (5).
Using the identity $g^{\mu\alpha}g_{\nu\alpha} = \delta^{\mu}_{\nu}$, we obtain

$$\frac{\partial g^{\mu\alpha}}{\partial x^\sigma}g_{\nu\alpha} = -g^{\mu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^\sigma}. \qquad (36)$$

Using the definition of the Christoffel symbols it is easy to show

$$\frac{\partial g_{\nu\alpha}}{\partial x^\sigma} = g_{\beta\alpha}\Gamma^\beta_{\nu\sigma} + g_{\beta\nu}\Gamma^\beta_{\alpha\sigma}. \tag{37}$$

Substituting this in (36) we obtain

$$\frac{\partial g^{\mu\alpha}}{\partial x^\sigma} g_{\nu\alpha} = -g^{\mu\alpha}(g_{\beta\alpha}\Gamma^\beta_{\nu\sigma} + g_{\beta\nu}\Gamma^\beta_{\alpha\sigma}). \tag{38}$$

If we multiply the last equation with $g^{\nu\gamma}$ and using $g^{\nu\gamma}g_{\nu\alpha} = \delta^\gamma_\alpha$ we have

$$\frac{\partial g^{\mu\gamma}}{\partial x^\sigma} = -g^{\mu\alpha}g_{\beta\alpha}g^{\nu\gamma}\Gamma^\beta_{\nu\sigma} - g^{\mu\alpha}g_{\beta\nu}g^{\nu\gamma}\Gamma^\beta_{\alpha\sigma} = -g^{\nu\gamma}\Gamma^\mu_{\nu\sigma} - g^{\mu\alpha}\Gamma^\gamma_{\alpha\sigma}. \tag{39}$$

Finally, if we replace ν with α and γ with ν in the last equation we obtain the equation (5).

In order to prove the equation (6) we use the equation (10)

$$g = g_{\mu\alpha}G^{(\mu,\alpha)}. \tag{40}$$

From this and using $\frac{\partial G^{(\mu,\alpha)}}{\partial g_{\mu\nu}} = 0$ and $\frac{\partial g_{\mu\alpha}}{\partial g_{\mu\nu}} = \delta^\nu_\alpha$ we have

$$\frac{\partial g}{\partial g_{\mu\nu}} = g_{\mu\alpha}\frac{\partial G^{(\mu,\alpha)}}{\partial g_{\mu\nu}} + G^{(\mu,\alpha)}\frac{\partial g_{\mu\alpha}}{\partial g_{\mu\nu}} = G^{(\mu,\nu)}, \tag{41}$$

$$\frac{\partial g}{\partial x^\mu} = \frac{\partial g}{\partial g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = G^{(\alpha,\beta)}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}. \tag{42}$$

Using $g^{\alpha\beta} = \frac{G^{(\alpha,\beta)}}{g}$ the last equation becomes

$$\frac{\partial g}{\partial x^\mu} = g g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}. \tag{43}$$

Substituting $\frac{\partial g_{\alpha\beta}}{\partial x^\mu}$ from (37) we obtain

$$\frac{\partial g}{\partial x^\mu} = g\Gamma^\alpha_{\alpha\mu} + g\Gamma^\beta_{\beta\mu} = 2g\Gamma^\alpha_{\alpha\mu}. \tag{44}$$

From this we obtain equation (6)

$$\Gamma^\alpha_{\alpha\mu} = \frac{\partial}{\partial x^\mu}\ln\sqrt{-g}. \tag{45}$$

In order to prove (7) we write

$$\begin{aligned}\Box\varphi &= \nabla_\mu\nabla^\mu\varphi = \nabla_\mu(g^{\mu\rho}\nabla_\rho\varphi) = \nabla_\mu(g^{\mu\rho}\frac{\partial\varphi}{\partial x^\rho}) \\ &= \frac{\partial}{\partial x^\mu}(g^{\mu\rho}\frac{\partial\varphi}{\partial x^\rho}) + \Gamma^\mu_{\mu\nu}g^{\nu\rho}\frac{\partial\varphi}{\partial x^\rho},\end{aligned} \tag{46}$$

where φ is any scalar function.

Using (6) we obtain

$$\begin{aligned}\Box\varphi &= \frac{\partial}{\partial x^\mu}(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}) + \frac{\partial}{\partial x^\nu}(\ln\sqrt{-g})g^{\nu\rho}\frac{\partial\varphi}{\partial x^\rho} \\ &= \frac{\partial}{\partial x^\mu}(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}) + \frac{\partial}{\partial x^\mu}(\ln\sqrt{-g})g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}. \end{aligned} \tag{47}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g}g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}) \\ &= \frac{1}{\sqrt{-g}}\Big(\frac{\partial}{\partial x^\mu}(\sqrt{-g})g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu} + \sqrt{-g}\frac{\partial g^{\mu\nu}}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} + \sqrt{-g}g^{\mu\nu}\frac{\partial^2\varphi}{\partial x^\mu\partial x^\nu}\Big) \\ &= \frac{\partial}{\partial x^\mu}(\ln\sqrt{-g})g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu} + \frac{\partial g^{\mu\nu}}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} + g^{\mu\nu}\frac{\partial^2\varphi}{\partial x^\mu\partial x^\nu} \\ &= \frac{\partial}{\partial x^\mu}(\ln\sqrt{-g})g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu} + \frac{\partial}{\partial x^\mu}(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}). \end{aligned} \tag{48}$$

From (47) and (48) we can conclude $\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi$ for any scalar function φ.

3. Derivation of equations of motion

Let's introduce the following actions

$$S_0 = \int (R-2\Lambda)\sqrt{-g}\; d^4x, \tag{49}$$

$$S_1 = \int \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\; d^4x. \tag{50}$$

then the variation of the action (1) can be expressed as

$$\delta S = \frac{1}{16\pi G}\delta S_0 + C\delta S_1. \tag{51}$$

3.1. The variation of S_0

The variation of S_0 can be found as follows

$$
\begin{aligned}
\delta S_0 &= \int \delta((R-2\Lambda)\sqrt{-g})\ d^4x \\
&= \int \delta(R\sqrt{-g})\ d^4x - 2\Lambda \int \delta\sqrt{-g}\ d^4x \\
&= \int (\sqrt{-g}\delta R + R\delta\sqrt{-g})\ d^4x + \Lambda \int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x \\
&= \int (\sqrt{-g}\delta(g^{\mu\nu}R_{\mu\nu}) - \frac{1}{2}R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu})\ d^4x \\
&+ \Lambda \int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x \\
&= \int R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x + \int g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu}\ d^4x \\
&- \frac{1}{2}\int Rg_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x + \Lambda \int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x \\
&= \int (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\sqrt{-g}\delta g^{\mu\nu}\ d^4x + \Lambda \int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x \\
&+ \int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\ d^4x = 0.
\end{aligned} \tag{52}
$$

Here we have used equation (3).
Now we want to show that $\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\ d^4x = 0$.
We first introduce

$$W^\nu = -g^{\mu\alpha}\delta\Gamma^\nu_{\mu\alpha} + g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha}. \tag{53}$$

We have

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu) = \frac{\partial W^\nu}{\partial x^\nu} + W^\nu \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\nu}. \tag{54}$$

Using equation (6) we get

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu) &= -\frac{\partial}{\partial x^\nu}(g^{\mu\alpha}\delta\Gamma^\nu_{\mu\alpha}) + \frac{\partial}{\partial x^\nu}(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha}) \\
&+ (-g^{\mu\alpha}\delta\Gamma^\nu_{\mu\alpha} + g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha})\Gamma^\beta_{\nu\beta} \\
&= -\frac{\partial g^{\mu\alpha}}{\partial x^\nu}\delta\Gamma^\nu_{\mu\alpha} - g^{\mu\alpha}\delta\frac{\partial\Gamma^\nu_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g^{\mu\nu}}{\partial x^\nu}\delta\Gamma^\alpha_{\mu\alpha} + g^{\mu\nu}\delta\frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} \\
&+ (-g^{\mu\alpha}\delta\Gamma^\nu_{\mu\alpha} + g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha})\Gamma^\beta_{\nu\beta}.
\end{aligned} \tag{55}
$$

Now using equation (5) we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu) &= g^{\alpha\beta}\Gamma^\mu_{\nu\beta}\delta\Gamma^\nu_{\mu\alpha} + g^{\mu\beta}\Gamma^\alpha_{\nu\beta}\delta\Gamma^\nu_{\mu\alpha} \\
-g^{\beta\nu}\Gamma^\mu_{\nu\beta}\delta\Gamma^\alpha_{\mu\alpha} - g^{\mu\beta}\Gamma^\nu_{\nu\beta}\delta\Gamma^\alpha_{\mu\alpha} &- g^{\mu\alpha}\delta\frac{\partial\Gamma^\nu_{\mu\alpha}}{\partial x^\nu} + g^{\mu\nu}\delta\frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} \\
&-g^{\mu\alpha}\Gamma^\beta_{\nu\beta}\delta\Gamma^\nu_{\mu\alpha} + g^{\mu\nu}\Gamma^\beta_{\nu\beta}\delta\Gamma^\alpha_{\mu\alpha} \\
&= g^{\mu\nu}\Big(- \delta\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \delta\frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} + \Gamma^\beta_{\alpha\mu}\delta\Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\nu\beta}\delta\Gamma^\beta_{\mu\alpha} \\
&\qquad -\Gamma^\beta_{\nu\mu}\delta\Gamma^\alpha_{\beta\alpha} - \Gamma^\alpha_{\beta\alpha}\delta\Gamma^\beta_{\mu\nu}\Big) \\
= g^{\mu\nu}\delta\Big(- \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} &+ \Gamma^\beta_{\alpha\mu}\Gamma^\alpha_{\beta\nu} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}\Big) = g^{\mu\nu}\delta R_{\mu\nu}. \qquad (56)
\end{aligned}
$$

Here we have relabeled some dummy indices.
Finally, we have

$$
g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu), \qquad (57)
$$

$$
\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x = \int \frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu)d^4x. \qquad (58)
$$

Using the Gauss-Stokes theorem we have

$$
\int \frac{\partial}{\partial x^\nu}(\sqrt{-g}W^\nu)d^4x = \int_\Sigma W^\nu d\sigma_\nu, \qquad (59)
$$

where Σ is boundary of a space-time region of integration.
Since $\delta g_{\mu\nu} = 0$ and $\delta(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}) = 0$ at the boundary of region of integration we have $W^\nu|_\Sigma = 0$. Then we have $\int_\Sigma W^\nu d\sigma_\nu = 0$ and finally

$$
\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x = 0. \qquad (60)
$$

Substituting this in (52) we obtain the variation of S_0

$$
\delta S_0 = \int G_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x + \Lambda\int g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x, \qquad (61)
$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

3.2. Preliminaries for the variation of S_1

1. Using the equation (4), for any scalar function h we have

$$\int h\delta R\sqrt{-g}\, d^4x = \int \left(hR_{\mu\nu}\delta g^{\mu\nu} + hg_{\mu\nu}\Box\delta g^{\mu\nu} - h\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\sqrt{-g}\, d^4x. \qquad (62)$$

The second and third term in this formula can be transformed in the following way:

$$\int hg_{\mu\nu}\Box\delta g^{\mu\nu}\sqrt{-g}\, d^4x = \int g_{\mu\nu}\Box h\delta g^{\mu\nu}\sqrt{-g}\, d^4x, \qquad (63)$$

$$\int h\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\sqrt{-g}\, d^4x = \int \nabla_\mu\nabla_\nu h\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x. \qquad (64)$$

To prove the first of these equations we use the Stokes's theorem and obtain

$$\begin{aligned}\int hg_{\mu\nu}\Box\delta g^{\mu\nu}\sqrt{-g}\, d^4x &= \int hg_{\mu\nu}\nabla_\alpha\nabla^\alpha\delta g^{\mu\nu}\sqrt{-g}\, d^4x\\ &= -\int \nabla_\alpha(hg_{\mu\nu})\nabla^\alpha\delta g^{\mu\nu}\sqrt{-g}\, d^4x\\ &= \int \nabla^\alpha\nabla_\alpha(hg_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g}\, d^4x\\ &= \int g_{\mu\nu}\nabla^\alpha\nabla_\alpha h\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x\\ &= \int g_{\mu\nu}\Box h\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x. \qquad (65)\end{aligned}$$

Here we have used $\nabla_\gamma g_{\mu\nu} = 0$ and $\nabla^\alpha\nabla_\alpha = \nabla_\alpha\nabla^\alpha = \Box$ to obtain the last integral.

To obtain the second equation we first introduce vector

$$N^\mu = h\nabla_\nu\delta g^{\mu\nu} - \nabla_\nu h\delta g^{\mu\nu}. \qquad (66)$$

From the above expression we have

$$\begin{aligned}\nabla_\mu N^\mu &= \nabla_\mu(h\nabla_\nu\delta g^{\mu\nu} - \nabla_\nu h\delta g^{\mu\nu})\\ &= \nabla_\mu h\nabla_\nu\delta g^{\mu\nu} + h\nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu h\, \delta g^{\mu\nu} - \nabla_\nu h\nabla_\mu\delta g^{\mu\nu}\\ &= h\nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu h\, \delta g^{\mu\nu}. \qquad (67)\end{aligned}$$

Integrating $\nabla_\mu N^\mu$ yields

$$\int \nabla_\mu N^\mu \sqrt{-g}\, d^4x = \int_\Sigma N^\mu n_\mu d\Sigma = 0, \tag{68}$$

where Σ is boundary of region of integration and n_μ is the unit normal vector. Since $N^\mu|_\Sigma = 0$ we have that the last integral is zero, which completes the proof.

Finally, we obtain:

$$\int h\delta R\sqrt{-g}\, d^4x = \int (hR_{\mu\nu} + g_{\mu\nu}\Box h - \nabla_\mu\nabla_\nu h)\,\delta g^{\mu\nu}\sqrt{-g}\, d^4x. \tag{69}$$

2. Let θ and ψ be scalar functions such that $\delta\psi|_\Sigma = 0$. Then we have

$$\begin{aligned}\int \theta\delta\Box\psi\sqrt{-g}\, d^4x &= \int \theta\partial_\alpha\delta(\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi)\, d^4x \\ &+ \int \theta\delta\left(\frac{1}{\sqrt{-g}}\right)\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi)\sqrt{-g}\, d^4x \\ &= \int \partial_\alpha(\theta\delta(\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi))\, d^4x - \int \partial_\alpha\theta\ \delta(\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi)\, d^4x \\ &+ \frac{1}{2}\int \theta g_{\mu\nu}\Box\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x.\end{aligned} \tag{70}$$

It is easy to see that $\int \partial_\alpha(\theta\delta(\sqrt{-g}g^{\alpha\beta}\partial_\beta\psi))\, d^4x = 0$. From this result it follows

$$\begin{aligned}&\int \theta\delta\Box\psi\sqrt{-g}\, d^4x \\ &= -\int g^{\alpha\beta}\partial_\alpha\theta\ \partial_\beta\psi\delta(\sqrt{-g})\, d^4x - \int \partial_\alpha\theta\ \partial_\beta\psi\delta g^{\alpha\beta}\sqrt{-g}\, d^4x \\ &- \int g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta\ \partial_\beta\delta\psi\, d^4x + \frac{1}{2}\int \theta g_{\mu\nu}\Box\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &= \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\ \partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\ \sqrt{-g}\, d^4x - \int \partial_\mu\theta\ \partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &- \int \partial_\beta(g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta\ \delta\psi)\, d^4x + \int \partial_\beta(g^{\alpha\beta}\sqrt{-g}\partial_\alpha\theta)\ \delta\psi\, d^4x \\ &+ \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &= \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\ \partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\ \sqrt{-g}\, d^4x - \int \partial_\mu\theta\ \partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &+ \int \Box\theta\ \delta\psi\sqrt{-g}\, d^4x + \frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\, d^4x.\end{aligned} \tag{71}$$

At the end we have that

$$\int \theta\delta\Box\psi\sqrt{-g}\; d^4x = \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\theta\;\partial_\beta\psi g_{\mu\nu}\delta g^{\mu\nu}\;\sqrt{-g}\; d^4x$$
$$-\int \partial_\mu\theta\;\partial_\nu\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x + \int \Box\theta\;\delta\psi\sqrt{-g}\; d^4x$$
$$+\frac{1}{2}\int g_{\mu\nu}\theta\Box\psi\delta g^{\mu\nu}\sqrt{-g}\; d^4x. \tag{72}$$

3.3. The variation of S_1

Now, after this preliminary work we can get the variation of S_1.

The variation of S_1 can be expressed as

$$\begin{aligned}\delta S_1 = &\int \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta(\sqrt{-g})\; d^4x\\ &+\int \delta(\mathcal{H}(R))\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\; d^4x\\ &+\int \mathcal{H}(R)\delta(\mathcal{F}(\Box)\mathcal{G}(R))\sqrt{-g}\; d^4x.\end{aligned} \tag{73}$$

Let us introduce notation

$$\begin{aligned}K_{\mu\nu} &= \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box,\\ \Phi &= \mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R) + \mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R),\end{aligned} \tag{74}$$

where $'$ denotes derivative with respect to R. For the first two integrals in the last equation we have

$$\begin{aligned}I_1 &= \int \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta(\sqrt{-g})\; d^4x\\ &= -\frac{1}{2}\int g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\; d^4x,\end{aligned} \tag{75}$$

$$I_2 = \int \delta(\mathcal{H}(R))\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\; d^4x = \int \mathcal{H}'(R)\delta R\;\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\; d^4x.$$

Substituting $h = \mathcal{H}'(R)\;\mathcal{F}(\Box)\mathcal{G}(R)$ in equation (69) we obtain

$$I_2 = \int \Big(R_{\mu\nu}\mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R) - K_{\mu\nu}\big(\mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R)\big)\Big)\delta g^{\mu\nu}\sqrt{-g}\; d^4x. \tag{76}$$

The third integral can be divided into linear combination of the following integrals

$$J_n = \int \mathcal{H}(R)\delta(\Box^n\mathcal{G}(R))\sqrt{-g}\; d^4x. \tag{77}$$

J_0 is the integral of the same form as I_2 so

$$J_0 = \int \Big(R_{\mu\nu}\mathcal{G}'(R)\mathcal{H}(R) - K_{\mu\nu}\big(\mathcal{G}'(R)\mathcal{H}(R)\big)\Big)\delta g^{\mu\nu}\sqrt{-g}\, d^4x. \tag{78}$$

For $n > 0$, we can find J_n using (72).
In the first step we take $\theta = \mathcal{H}(R)$ and $\psi = \Box^{n-1}\mathcal{G}(R)$ and obtain

$$\begin{aligned} J_n = &\frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\mathcal{H}(R)\, \partial_\beta\Box^{n-1}\mathcal{G}(R) g_{\mu\nu}\delta g^{\mu\nu}\, \sqrt{-g}\, d^4x \\ &- \int \partial_\mu\mathcal{H}(R)\, \partial_\nu\Box^{n-1}\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \int \Box\mathcal{H}(R)\, \delta\Box^{n-1}\mathcal{G}(R)\sqrt{-g}\, d^4x \\ &+ \frac{1}{2}\int g_{\mu\nu}\mathcal{H}(R)\Box^n\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x. \end{aligned} \tag{79}$$

In the second step we take $\theta = \Box\mathcal{H}(R)$ and $\psi = \Box^{n-2}\mathcal{G}(R)$ and get the third integral in this formula, etc. Using (72) n times we obtain

$$\begin{aligned} J_n = &\frac{1}{2}\sum_{l=0}^{n-1}\int g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Box^l\mathcal{H}(R)\partial_\beta\Box^{n-1-l}\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &- \sum_{l=0}^{n-1}\int \partial_\mu\Box^l\mathcal{H}(R)\partial_\nu\Box^{n-1-l}\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &+ \frac{1}{2}\sum_{l=0}^{n-1}\int g_{\mu\nu}\Box^l\mathcal{H}(R)\Box^{n-l}\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &+ \int \Big(R_{\mu\nu}\mathcal{G}'(R)\Box^n\mathcal{H}(R) - K_{\mu\nu}\big(\mathcal{G}'(R)\Box^n\mathcal{H}(R)\big)\Big)\delta g^{\mu\nu}\sqrt{-g}\, d^4x. \end{aligned} \tag{80}$$

Using the equation (69) we obtain the last integral in the above formula.
Finally, we can put everything together and obtain

$$\begin{aligned} \delta S_1 = &I_1 + I_2 + \sum_{n=0}^{\infty} f_n J_n \\ = &-\frac{1}{2}\int g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &+ \int \left(R_{\mu\nu}\Phi - K_{\mu\nu}\Phi\right)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \\ &+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Box^l\mathcal{H}(R)\partial_\beta\Box^{n-1-l}\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\, d^4x \end{aligned}$$

$$- \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \int \partial_\mu \Box^l \mathcal{H}(R) \partial_\nu \Box^{n-1-l} \mathcal{G}(R) \delta g^{\mu\nu} \sqrt{-g}\, d^4x$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \int g_{\mu\nu} \Box^l \mathcal{H}(R) \Box^{n-l} \mathcal{G}(R) \delta g^{\mu\nu} \sqrt{-g}\, d^4x. \tag{81}$$

3.4. The variation of S and the equations of motion

We have

$$\delta S = \int \frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} \delta g^{\mu\nu} \sqrt{-g}\, d^4x + C\delta S_1. \tag{82}$$

From $\delta S = 0$ we obtain the equations of motion (EOM). We can write the EOM in the form

$$\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} + C\Big(- \frac{1}{2} g_{\mu\nu} \mathcal{H}(R) \mathcal{F}(\Box) \mathcal{G}(R) + (R_{\mu\nu}\Phi - K_{\mu\nu}\Phi)$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Box^l \mathcal{H}(R) \partial_\beta \Box^{n-1-l} \mathcal{G}(R)$$
$$- 2\partial_\mu \Box^l \mathcal{H}(R) \partial_\nu \Box^{n-1-l} \mathcal{G}(R) + g_{\mu\nu} \Box^l \mathcal{H}(R) \Box^{n-l} \mathcal{G}(R)\big)\Big) = 0, \tag{83}$$

where $K_{\mu\nu}$ and Φ are given by expressions (74).

In the case when the metric is homogeneous and isotropic, i.e. Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the last equation is equivalent to the following system (trace and 00 component):

$$\frac{4\Lambda - R}{16\pi G} + C\Big(- 2\mathcal{H}(R) \mathcal{F}(\Box) \mathcal{G}(R) + (R\Phi + 3\Box\Phi) \tag{84}$$
$$+ \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(\partial_\mu \Box^l \mathcal{H}(R) \partial^\mu \Box^{n-1-l} \mathcal{G}(R) + 2\Box^l \mathcal{H}(R) \Box^{n-l} \mathcal{G}(R)\big)\Big) = 0,$$

$$\frac{G_{00} + \Lambda g_{00}}{16\pi G} + C\Big(- \frac{1}{2} g_{00} \mathcal{H}(R) \mathcal{F}(\Box) \mathcal{G}(R) + (R_{00}\Phi - K_{00}\Phi)$$
$$+ \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(g_{00} g^{\alpha\beta} \partial_\alpha \Box^l \mathcal{H}(R) \partial_\beta \Box^{n-1-l} \mathcal{G}(R)$$
$$- 2\partial_0 \Box^l \mathcal{H}(R) \partial_0 \Box^{n-1-l} \mathcal{G}(R) + g_{00} \Box^l \mathcal{H}(R) \Box^{n-l} \mathcal{G}(R)\big)\Big) = 0. \tag{85}$$

4. Conclusion

In this paper we have considered a nonlocal gravity model without matter given by the action in the form

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + C\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\Big). \tag{86}$$

We have derived the equations of motion for this action. In many research papers there are equations of motion which are a special case of our equations. In the case $\mathcal{H}(R) = \mathcal{G}(R) = R$ we obtain

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + CR\mathcal{F}(\Box)R\Big). \tag{87}$$

Studies of this model can be found in [2, 3, 4, 8]. The case $\mathcal{H}(R) = R^{-1}$ and $\mathcal{G}(R) = R$ we analyze in [10].

Investigation of equations of motion and finding its solutions is a very difficult task. Using ansätze we can simplify the problem and get some solutions. For some examples of ansätze see [7, 9].

Acknowledgements

This investigation is supported by Ministry of Education and Science of the Republic of Serbia, grant No 174012. I would like to thank Branko Dragovich, Zoran Rakić and Ivan Dimitrijević for useful discussions.

References

[1] S. Nojiri, S. D. Odintsov, *Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models*, Phys. Rept. **505** (2011) 59–144 [arXiv:1011.0544v4 [gr-qc]].

[2] T. Biswas, T. Koivisto, A. Mazumdar, *Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity*, JCAP **1011** (2010) 008 [arXiv:1005.0590v2 [hep-th]].

[3] A. S. Koshelev, S. Yu. Vernov, *On bouncing solutions in non-local gravity*, Phys. Part. Nuclei **43**, 666668 (2012) [arXiv:1202.1289v1 [hep-th]].

[4] T. Biswas, A. S. Koshelev, A. Mazumdar, S. Yu. Vernov, *Stable bounce and inflation in non-local higher derivative cosmology*, JCAP **08** (2012) 024, [arXiv:1206.6374v2 [astro-ph.CO]].

[5] M. Pantic, *Introduction to Einstein's theory of gravity* (Novi Sad, 2005, in Serbian).

[6] A. Guarnizo, L. Castaneda, J. M. Tejeiro, *Boundary Term in Metric $f(R)$ Gravity: Field Equations in the Metric Formalism*, (2010) [arXiv:1002.0617v4 [gr-qc]].

[7] I. Dimitrijevic, B. Dragovich, J. Grujic, Z. Rakic, *On modified gravity*, Springer Proceedings in Mathematics and Statistics **36** (2013) [arXiv:1202.2352v2 [hep-th]].

[8] I. Dimitrijevic, B. Dragovich, J. Grujic, Z. Rakic, *New cosmological solutions in nonlocal modified gravity*, to appear in Romanian Journal of Physics [arXiv:1302.2794 [gr-qc]].

[9] I. Dimitrijevic, *Some ansätze in nonlocal modified gravity*, in these proceedings.

[10] I. Dimitrijevic, B. Dragovich, J. Grujic and Z. Rakic, to be published.