Noncommutative gravity and the Seiberg-Witten map *

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Abstract

We discus the Seiberg-Witten map and its application to noncommutative gauge theories. In particular, we present the method of composite fields which we find very useful when calculating higher order corrections in noncommutative gauge theories. Two examples are given: one is the calculation of the second order correction for the noncommutative Yang-Mills action and the other is the calculation of the corrections for the AdS inspired noncommutative gravity action. The second example we discuss in more details.

1. Introduction

Field theories on noncommutative (NC) spaces have been investigated in many aspects during the last twenty years. Various approaches to definition and analysis of the properties of noncommutative spaces are present in the literature [1]. One of the the most frequently used is the approach of deformation quantization [2]. In this approach noncommutative functions

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* The work of the authors is supported by the project 171031 of the Serbian Ministry of Education and Science.

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$\hat{f}(\hat{x})$ are mapped to the functions of commuting coordinates $f(x)$ and the abstract algebra multiplication is represented by a $\star$-product, which is a deformation of the usual point-wise multiplication. The simplest and the most analyzed example of the $\star$-product is the Moyal-Weyl $\star$-product

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x} \ , \qquad (1)$$

defined by a constant antisymmetric matrix $\theta^{\mu\nu}$. Using this type of deformation various problems were investigated: NC scalar field theories, NC gauge theories, deformations of supersymmetric theories,....

An important boost to the formulation of NC gauge theories came with the paper of Seiberg and Witten [3]. They found a connection between gauge theories on the commutative space and the corresponding NC gauge theories. This result was then used by Wess et al. [4] to formulate the enveloping algebra approach to NC gauge theories. Namely, for groups which are of importance for physical applications like $SU(N)$, NC gauge transformations only close in the enveloping algebra which is infinitely dimensional. This implies that the NC gauge field is also enveloping algebra-valued which leads to infinitely many new degrees of freedom. However, using the Seiberg-Witten (SW) map one can express all enveloping algebra-valued NC variables (gauge parameter and fields) in terms of the corresponding commutative variables. In that way both commutative and noncommutative theory have the same number of degrees of freedom. This approach enabled the analysis of renormalizability of NC gauge theories [5], construction of a NC deformation of the Standard Model [6] and investigation of its phenomenological consequences [7]. On the other hand, construction of a NC generalization of General Relativity (GR) proved to be a difficult task. Having in mind that the SW approach works very well for NC gauge theories, many authors consider NC gravity as a gauge theory of the Lorentz/Poincaré group, [8]. It was shown there that if reality of the NC gravity action is imposed, all odd order corrections (in the NC parameter) have to vanish. The first non-vanishing correction is then the second order correction.

In this paper we review the SW map and outline the method of composite fields. Then we use this method to calculate the second order correction of the NC Yang-Mills action. Finally, we discuss the NC gravity theory based on the MacDowell Mansouri action [9] and present our results.

2. Noncommutative gauge theories

The simplest and the most studied form of noncommutativity is the canonical or θ-constant noncommutativity, given by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \qquad (2)$$

with the constant antisymmetric matrix $\theta^{\mu\nu}$. Following the approach of deformation quantization we represent noncommutative functions as functions of commuting coordinates and algebra multiplication with the Moyal-Weyl $\star$-product (1)

$$\hat{f}(\hat{x}) \mapsto f(x)$$
$$\hat{f}(\hat{x})\hat{g}(\hat{x}) \mapsto (f \star g)(x).$$

The infinitesimal noncommutative gauge transformations are now defined as [4]

$$\delta^\star \psi = i\Lambda \star \psi(x), \tag{3}$$

where Λ is the noncommutative gauge parameter and ψ is the noncommutative matter field. Before proceeding to the standard construction one should check if these transformations close in the algebra. If the noncommutative gauge parameter Λ is supposed to be Lie algebra-valued $\Lambda(x) = \Lambda^a(x)T^a$, the explicit calculation gives

$$\begin{aligned}(\delta_1^\star\delta_2^\star - \delta_2^\star\delta_1^\star)\psi(x) &= (\Lambda_1 \star \Lambda_2 - \Lambda_2 \star \Lambda_1) \star \psi \\ = \frac{1}{2}\Big([\Lambda_1^a \stackrel{\star}{,} \Lambda_2^b]\{T^a, T^b\} &+ \{\Lambda_1^a \stackrel{\star}{,} \Lambda_2^b\}[T^a, T^b]\Big) \star \psi.\end{aligned} \tag{4}$$

The left hand side of (4) in general does not close because of the first term in the last line. Namely, an anticommutator of two generators is in general no longer in the Lie algebra of generators. There are two ways of solving this problem. One is to consider only $U(N)$ gauge theories since then the anticommutator of generators is still in the Lie algebra of generators. This approach enables studying of non-expanded (in orders of the deformation parameter) noncommutative field theories. Quantizing these theories leads to mixing of ultraviolet (UV) and infrared (IR) divergences which is know in the literature as UV/IR mixing [10].

The second possibility is to use the enveloping algebra approach [4]. The NC gauge parameter Λ_α is said to be enveloping algebra-valued and in that case the algebra (4) closes. However, the NC gauge field also has to be enveloping algebra-valued,

$$A_\mu = A_\mu^{(0)a}T^a + \frac{1}{2}A_\mu^{ab}\{T^a, T^b\} + \dots$$

and it seems as we obtained a theory with infinitely many degrees of freedom, since the enveloping algebra is infinitely dimensional. The solution to this problem and is given in terms of the Seiberg-Witten (SW)-map [3].

The basic assumption of the SW-map is that the noncommutative fields and the noncommutative gauge parameter can be expressed as functions of the commutative fields and the commutative gauge parameter α. For example, the noncommutative gauge parameter Λ is given by

$$\Lambda = \Lambda(\alpha, A_\mu^{(0)}) := \Lambda_\alpha(A_\mu^{(0)}) \tag{5}$$

with the commutative gauge field $A_\mu^{(0)}$ and $\delta_\alpha A_\mu^{(0)} = \partial_\mu\alpha + i[\alpha, A_\mu^{(0)}]$. The explicit form of this dependence is found by solving the appropriate equations. In this way the number of degrees of freedom in the noncommutative theory reduces to the number of degrees of freedom of the corresponding commutative theory.

The requirement that the commutator of two NC gauge transformations is a NC gauge transformation again (4) gives the solution for $\Lambda_\alpha^{(1)}, \Lambda_\alpha^{(2)}, \dots$. The recursive relation between the $(n+1)$st order and the nth order solution is given by

$$\Lambda_\alpha^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{A_\kappa \stackrel{\star}{,} \partial_\lambda \Lambda_\alpha\}\Big)^{(n)} \ , \tag{6}$$

where $(A \star B)^{(n)} = A^{(n)}B^{(0)} + A^{(n-1)}B^{(1)} + \dots + A^{(0)} \star^{(1)} B^{(n-1)} + A^{(1)} \star^{(1)} B^{(n-2)} + \dots$ includes all possible terms of order n. The solution for the NC gauge field follows from

$$\delta_\alpha^\star A_\mu = \partial_\mu \Lambda_\alpha + i[\Lambda_\alpha \stackrel{\star}{,} A_\mu]. \tag{7}$$

The recursive solution in this case is given by

$$A_\mu^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{A_\kappa \stackrel{\star}{,} \partial_\lambda A_\mu + F_{\lambda\mu}\}\Big)^{(n)}. \tag{8}$$

Here $F_{\mu\nu}$ is the NC field strength tensor defined by $\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]$ and $\delta_\alpha^\star F_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\mu\nu}]$. Using these relations one can find the recursive solution for $F_{\mu\nu}$:

$$\begin{aligned} F_{\mu\nu}^{(n+1)} = &-\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{A_\kappa \stackrel{\star}{,} \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu}\}\Big)^{(n)} \\ &+\frac{1}{2(n+1)}\theta^{\kappa\lambda}\Big(\{F_{\mu\kappa}, \stackrel{\star}{,} F_{\nu\lambda}\}\Big)^{(n)} \end{aligned} \tag{9}$$

where $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - i[A_\lambda \stackrel{\star}{,} F_{\mu\nu}]$. One can also find the SW-map solutions for the matter fields transforming in the fundamental or adjoint representation, see [11]. For example, the field Φ in the adjoint representation is

$$\delta_\alpha^\star \Phi = i[\Lambda_\alpha \stackrel{\star}{,} \Phi], \tag{10}$$

$$\Phi^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{A_\kappa \stackrel{\star}{,} \partial_\lambda \Phi + D_\lambda \Phi\}\Big)^{(n)}, \tag{11}$$

with $D_\lambda \Phi = \partial_\lambda \Phi - i[\hat{\omega}_\lambda \stackrel{\star}{,} \Phi]$.

2.1. Method of composite fields

Let us calculate the $\star$-product of two fields in the adjoint representation, Φ_1 and Φ_2. It is given by

$$\Phi_1 \star \Phi_2 = \Phi_1^{(0)}\Phi_2^{(0)} + (\Phi_1 \star \Phi_2)^{(1)} + (\Phi_1 \star \Phi_2)^{(2)} + \dots \ . \tag{12}$$

Using (10) it is straightforward to check that this product transforms in adjoint representation of the gauge group. The first order of (12)

$$(\Phi_1 \star \Phi_2)^{(1)} = \Phi_1^{(1)}\Phi_2^{(0)} + \Phi_1^{(0)}\Phi_2^{(1)} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\Phi_1^{(0)}\partial_\beta\Phi_2^{(0)}$$

can be rewritten in the following form

$$\begin{aligned}(\Phi_1 \star \Phi_2)^{(1)} = &-\frac{1}{4}\theta^{\alpha\beta}\{A_\alpha^{(0)}, \partial_\beta(\Phi_1^{(0)}\Phi_2^{(0)}) + D_\beta(\Phi_1^{(0)}\Phi_2^{(0)})\} \\ &+\frac{i}{2}\theta^{\alpha\beta}(D_\alpha\Phi_1^{(0)})(D_\beta\Phi_2^{(0)}).\end{aligned} \tag{13}$$

The calculation is straightforward. Notice that the first term in (13) is a solution of the SW map for the field $\psi = \Phi_1 \star \Phi_2$ in the adjoint representation, compare with (11). The second term appears because the field $\psi = \Phi_1 \star \Phi_2$ is not a fundamental field but a product of two fundamental fields. Also notice that the second term is written in terms of covariant derivatives. This will be a big advantage when we write a NC action in a manifestly gauge covariant form.

One can generalize (13) and write an expression that is valid to all orders. We will not do that here, for details look at [12].

3. Example I: Noncommutative Yang-Mills action

The NC Yang-Mills action is defined as

$$S_{YM} = -\frac{1}{4}\int \mathrm{d}x^4 \mathrm{Tr}(F_{\mu\nu} \star F^{\mu\nu}). \tag{14}$$

Using the cyclicity of the integral[1] one can show that the action (14) is invariant under the NC gauge transformations. We would like to calculate the second order expansion of this action. Using (9) and the method

[1] The integral is cyclic if

$$\int \mathrm{d}x^4 f \star g = \int \mathrm{d}x^4 g \star f = \int \mathrm{d}x^4 fg.$$

described in the previous section, for the first order of the Lagrangian we obtain

$$
\begin{aligned}
(F_{\mu\nu} \star F^{\mu\nu})^{(1)} = & -\frac{1}{4}\theta^{\alpha\beta}\{A^{(0)}_\alpha, \partial_\beta(F^{(0)}_{\mu\nu}F^{\mu\nu(0)}) + D_\beta(F^{(0)}_{\mu\nu}F^{\mu\nu(0)})\} \\
& + \frac{i}{2}\theta^{\alpha\beta}(D_\alpha F^{(0)}_{\mu\nu})(D_\beta F^{\mu\nu(0)}) \\
& + \frac{1}{2}\theta^{\alpha\beta}\{F^{(0)}_{\alpha\mu}, F^{(0)}_{\beta\nu}\}F^{\mu\nu(0)} + \frac{1}{2}\theta^{\alpha\beta}F^{\mu\nu(0)}\{F^{(0)}_{\alpha\mu}, F^{(0)}_{\beta\nu}\}.
\end{aligned}
$$

The the first order of the action (14) is then

$$
S^{(1)}_{YM} = -\frac{1}{4}\theta^{\alpha\beta}\int \mathrm{d}x^4 \mathrm{Tr}\Big(2F^{\mu\nu(0)}F^{(0)}_{\alpha\mu}F^{(0)}_{\beta\nu} - \frac{1}{2}F^{(0)}_{\alpha\beta}F^{(0)}_{\mu\nu}F^{\mu\nu(0)}\Big). \tag{15}
$$

We used $[D_\alpha, D_\beta]F^{(0)}_{\mu\nu} = -i[F^{(0)}_{\alpha\beta}, F^{(0)}_{\mu\nu}]$. Now, we conjecture the recursive relation

$$
S^{(n+1)}_{YM} = -\frac{1}{4}\theta^{\alpha\beta}\int \mathrm{d}x^4 \mathrm{Tr}\Big(2F^{\mu\nu} \star F_{\alpha\mu} \star F_{\beta\nu} - \frac{1}{2}F_{\alpha\beta} \star F_{\mu\nu} \star F^{\mu\nu}\Big)^{(n)} \tag{16}
$$

and calculate the second order expansion from it. We find

$$
\begin{aligned}
S^{(2)}_{YM} = & -\frac{1}{8}\theta^{\alpha\beta}\theta^{\kappa\lambda}\int \mathrm{d}x^4 \mathrm{Tr}\Big\{F^{\mu\nu(0)}\Big(\{F^{(0)}_{\kappa\alpha}, F^{(0)}_{\lambda\mu}\}F^{(0)}_{\beta\nu} + F^{(0)}_{\alpha\mu}\{F^{(0)}_{\kappa\beta}, F^{(0)}_{\lambda\nu}\} \\
& - \frac{1}{4}\{F^{(0)}_{\kappa\alpha}, F^{(0)}_{\lambda\beta}\}F^{(0)}_{\mu\nu} - \frac{1}{2}\{F^{(0)}_{\kappa\lambda}, F^{(0)}_{\alpha\mu}F^{(0)}_{\beta\nu}\} + i(D_\kappa F^{(0)}_{\alpha\mu})(D_\lambda F^{(0)}_{\beta\nu})\Big) \\
& + F^{(0)}_{\alpha\beta}\Big(\frac{1}{8}\{F^{(0)}_{\kappa\lambda}, F^{(0)}_{\mu\nu}F^{\mu\nu(0)}\} - \frac{1}{2}\{F^{\mu\nu(0)}, F^{(0)}_{\kappa\mu}F^{(0)}_{\lambda\nu}\} \\
& - \frac{i}{4}(D_\kappa F^{(0)}_{\mu\nu})(D_\lambda F^{(\mu\nu 0)})\Big) + \{F^{(0)}_{\kappa\mu}, F^{(0)}_{\lambda\nu}\}F_\alpha{}^{\mu(0)}F_\beta{}^{\nu(0)}\Big)\Big\}.
\end{aligned} \tag{17}
$$

Note that the gauge invariance of this expression is obvious, all terms are written as functions of $F^{(0)}_{\mu\nu}$ and its covariant derivatives. If one tries to calculate the second order expansion of the action (14) just by straightforwardly $\star$-multiplying and inserting the first and the second order solutions, one would end up with a very complicated expression. In addition, that expression would not be written in a manifestly gauge invariant form, since the gauge field and its partial derivatives appear explicitly in (9). This is the reason we find the method of composite fields very useful.

4. Example II: AdS inspired noncommutative gravity

Let us now discuss a NC gravity theory based on the MacDowell-Mansouri action. The commutative action is given by

$$
\begin{aligned}
S &= \frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4x \mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}\gamma_5) \\
&= -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} + eR + 2e\Lambda\Big]\,.
\end{aligned} \tag{18}
$$

This aciton is invariant under the action of the gauge group $SO(1,3)$. The generators of $SO(1,3)$ are $\sigma_{ab}/2 = i/4[\gamma_a, \gamma_b]$, with γ_a the four dimensional Dirac gama matrices. The fields in the theory are the spin connection ω_μ and the vielbein e_μ:

$$\begin{aligned} \omega_\mu = \omega_\mu^{ab}\frac{\sigma_{ab}}{4}, \quad e_\mu = e_\mu^a\frac{\gamma_a}{2}, \\ \delta_\epsilon\omega_\mu = \partial_\mu\epsilon + i[\epsilon, \omega_\mu], \quad \delta_\epsilon e_\mu = i[\epsilon, e_\mu]. \end{aligned} \tag{19}$$

The gauge parameter ϵ is valued in the $SO(1,3)$ algebra, $\epsilon = \epsilon_\mu^{ab}\frac{\sigma_{ab}}{4}$. The field-strength tensor is the curvature tensor and is given by

$$R_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_m, \omega_\nu] = R_{\mu\nu}^{\ ab}\frac{\sigma_{ab}}{4}. \tag{20}$$

The first term in the action (18) is the topological Gauss-Bonnet term, the second term is the Einstein-Hilbert action and the last term is the cosmological constant. The determinant of the vielbein is $e = det(e_\mu^a)$.

The NC generalization of (18) is given by

$$\begin{aligned} S = \frac{il^2}{64\pi G_N}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\Big[\mathrm{Tr}(\hat{R}_{\mu\nu}\star\hat{R}_{\rho\sigma}\gamma_5) \\ -\frac{i}{l^2}\mathrm{Tr}(\hat{R}_{\mu\nu}\star\hat{E}_\rho\star\hat{E}_\sigma\gamma_5) - \frac{1}{4l^4}\mathrm{Tr}(\hat{E}_\mu\star\hat{E}_\nu\star\hat{E}_\rho\star\hat{E}_\sigma\gamma_5)\Big], \end{aligned} \tag{21}$$

with noncommutative vielbeins $\hat{E}_\mu$ and noncommutative curvature $\hat{R}_{\mu\nu}$ defined by

$$\hat{R}_{\mu\nu} = \partial_\mu\hat{\omega}_\nu - \partial_\nu\hat{\omega}_\mu - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\omega}_\nu], \tag{22}$$

where $\hat{\omega}_\mu$ is the noncommutative $SO(1,3)_\star$ gauge potential. Note that variables with "hat" are NC variables and we use SW-map solutions form the previous section to write them in terms of the commutative fields. The symmetry of the action (21) is the NC $SO(1,3)$ gauge symmetry. The SW-map guaranties that the expanded action is invariant under the commutative $SO(1,3)$.

We will analyze only the Einstein-Hilbert action in details. The analysis of the remaining two terms can be found in [9]. The Einstein-Hilbert action is given by

$$S_{EH} = \frac{1}{64\pi G_N}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\hat{R}_{\mu\nu}\star\hat{E}_\rho\star\hat{E}_\sigma\gamma_5). \tag{23}$$

To find the first order correction we first calculate $\hat{R}_{\mu\nu}\star\hat{E}_\rho\star\hat{E}_\sigma$. We consider $\hat{R}_{\mu\nu}\star\hat{E}_\rho\star\hat{E}_\sigma$ as a $\star$-product of the curvature tensor $\hat{R}_{\mu\nu}$ and the composite field $\hat{E}_\rho \star \hat{E}_\sigma$. Then

$$\begin{aligned}
(\hat{R}_{\mu\nu} \star \hat{E}_\rho \star \hat{E}_\sigma)^{(1)} &= \hat{R}^{(1)}_{\mu\nu}(e_\rho e_\sigma) + R_{\mu\nu}(\hat{E}_\rho \star \hat{E}_\sigma)^{(1)} \\
&\quad + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(R_{\mu\nu})\partial_\beta(e_\rho e_\sigma) \\
&= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta(R_{\mu\nu}e_\rho e_\sigma) + D_\beta(R_{\mu\nu}e_\rho e_\sigma)\} \\
&\quad + \frac{i}{2}\theta^{\alpha\beta}(D_\alpha R_{\mu\nu})D_\beta(e_\rho e_\sigma) \qquad (24) \\
&\quad + \frac{1}{2}\theta^{\alpha\beta}\{R_{\alpha\mu}, R_{\beta\nu}\}e_\rho e_\sigma + \frac{i}{2}\theta^{\alpha\beta}R_{\mu\nu}(D_\alpha e_\rho)(D_\beta e_\sigma)\ .
\end{aligned}$$

The first order correction of Einstein-Hilbert action is

$$S^{(1)}_{EH} = \frac{1}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4x \mathrm{Tr}\Big(\hat{R}_{\mu\nu} \star (\hat{E}_\rho \star \hat{E}_\sigma)\gamma_5\Big)^{(1)}\ . \qquad (25)$$

Inserting (24) in (25) and integrating by parts we obtain

$$\begin{aligned}
S^{(1)}_{EH} &= -\frac{1}{256\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\int \mathrm{d}^4x \mathrm{Tr}\gamma_5\Big(\{R_{\alpha\beta}, R_{\mu\nu}\}e_\rho e_\sigma \\
&\quad -2\{R_{\alpha\mu}, R_{\beta\nu}\}e_\rho e_\sigma - 2iR_{\mu\nu}(D_\alpha e_\rho)(D_\beta e_\sigma)\Big)\ . \qquad (26)
\end{aligned}$$

The second order correction of the Einstein-Hilbert action follows from the first order correction (26) as

$$\begin{aligned}
S^{(2)}_{EH} &= -\frac{1}{512\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\int \mathrm{d}^4x \mathrm{Tr}\gamma_5\Big(\{\hat{R}_{\alpha\beta} \stackrel{\star}{,} \hat{R}_{\mu\nu}\} \star \hat{E}_\rho \star \hat{E}_\sigma \qquad (27) \\
&\quad -2\{\hat{R}_{\alpha\mu} \stackrel{\star}{,} \hat{R}_{\beta\nu}\} \star \hat{E}_\rho \star \hat{E}_\sigma - 2i\hat{R}_{\mu\nu} \star (D_\alpha\hat{E}_\rho) \star (D_\beta\hat{E}_\sigma)\Big)^{(1)}.
\end{aligned}$$

Applying

$$\begin{aligned}
(\hat{R}_{\alpha\beta} \star \hat{R}_{\mu\nu})^{(1)} &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(R_{\alpha\beta}R_{\mu\nu}) + D_\lambda(R_{\alpha\beta}R_{\mu\nu})\} \\
&\quad + \frac{i}{2}\theta^{\kappa\lambda}(D_\kappa R_{\alpha\beta})(D_\lambda R_{\mu\nu}) + \frac{1}{2}\theta^{\kappa\lambda}(\{R_{\kappa\alpha}, R_{\lambda\beta}\}R_{\mu\nu} \\
&\quad + R_{\alpha\beta}\{R_{\kappa\mu,R_{\lambda\nu}}\})
\end{aligned}$$

and

$$(D_\alpha\hat{E}_\rho)^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(D_\alpha e_\rho) + D_\lambda(D_\alpha e_\rho)\} + \frac{1}{2}\theta^{\kappa\lambda}\{R_{\kappa\alpha}, D_\lambda e_\rho\},$$

$$(D_\alpha\hat{E}_\rho \star D_\beta\hat{E}_\sigma)^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(D_\alpha e_\rho D_\beta e_\sigma) + D_\lambda(D_\alpha e_\rho D_\beta e_\sigma)\} + \frac{i}{2}\theta^{\kappa\lambda}(D_\kappa D_\alpha e_\rho)(D_\lambda D_\beta e_\sigma) + \frac{1}{2}\theta^{\kappa\lambda}\Big(\{R_{\kappa\alpha}, D_\lambda e_\rho\}(D_\beta e_\sigma) + (D_\alpha e_\rho)\{R_{\kappa\beta}, D_\lambda e_\sigma\}\Big)$$

we obtain

$$S^{(2)}_{EH} = -\frac{1}{512\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\theta^{\kappa\lambda}\int \mathrm{d}^4x \mathrm{Tr}\gamma_5\Big((-\frac{1}{4}\{R_{\kappa\lambda}, \{R_{\alpha\beta}, R_{\mu\nu}\}\} + \{R_{\kappa\lambda}, \{R_{\alpha\mu}, R_{\beta\nu}\}\} + \frac{1}{2}\{R_{\mu\nu}, \{R_{\kappa\alpha}, R_{\lambda\beta}\}\} - 2\{R_{\alpha\mu}, \{R_{\kappa\beta}, R_{\lambda\nu}\}\} + \frac{i}{2}[D_\kappa R_{\alpha\beta}, D_\lambda R_{\mu\nu}] - i[D_\kappa R_{\alpha\mu}, D_\lambda R_{\beta\nu}])e_\rho e_\sigma + i(\{R_{\alpha\beta}, R_{\mu\nu}\} - 2\{R_{\alpha\mu}, R_{\beta\nu}\})(D_\kappa e_\rho)(D_\lambda e_\sigma) - iR_{\mu\nu}\{D_\alpha e_\rho, \{R_{\kappa\beta}, D_\lambda e_\sigma\}\} + R_{\mu\nu}(D_\kappa D_\alpha e_\rho)(D_\lambda D_\beta e_\sigma)\Big) . \quad (28)$$

The second order correction to the Einstein-Hilbert action is of the 3rd, 2nd and 1st order in the curvature. Its implications to the equations of motion have to be investigated carefully. Since the higher powers of curvature enter, it is obvious that unlike in the General Relativity the spin connection propagates and it cannot be expressed in terms of vielbeins.

5. Discussion and outlook

We presented a way to calculate the higher order corrections for the NC gauge theories expanded in terms of the SW-map. Two examples are given: the second order correction of the NC Yang-Mills action and the second order correction of the NC gravity action. Both corrections are written in a manifestly gauge covariant way. The consequences: equations of motion, corrections to the commutative solutions, phenomenology,... remain to be studied in the future work.

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