Nontrivial Kalb-Ramond field of the effective non-geometric background *

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Abstract

We solve the boundary conditions for the open bosonic string moving in the weakly curved background. This background is composed of the constant metric and linear in coordinate Kalb-Ramond field, with the infinitesimal coordinate dependence. The effective theory obtained on the above solution is defined on the non-geometric doubled space $(q^\mu, \tilde{q}_\mu)$, where q^μ is the effective coordinate and $\tilde{q}_\mu$ is its T-dual. The effective metric depends on the coordinate q^μ and there exists the effective Kalb-Ramond field which depends on the T-dual coordinate $\tilde{q}_\mu$.

1. Introduction

In a large number of papers, considering the open bosonic string theory, is assumed that the string moves in the flat background. In this case, the equations of motion and the boundary conditions, obtained from the minimal action principle, can be solved by expressing the odd coordinate part in terms of the even coordinate part. We investigated a generalization of such a solution, when the string moves in the weakly curved background.

The weakly curved background is composed of the constant metric and the linearly coordinate dependent Kalb-Ramond field, with infinitesimally small field strength. We sought for the solution of the boundary conditions in three ways. In our paper [1], we applied the Dirac consistency procedure to the boundary conditions and obtained the infinite set of constraints. We gathered them into two sigma dependent constraints, and obtained theirs explicit form and solved them iteratively. As we used the canonical approach, the boundary conditions where presented in the canonical form,

* Work supported in part by the Serbian Ministry of Education, Science and Technological Development, under contract No. 171031.

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and their solution gave the odd coordinate and momenta parts in terms of the even ones.

In our paper [2], we repeated the Dirac procedure, but this time completely in the Lagrangian formalism. The solution gives the sigma and time derivative of the odd coordinate part in therms of the even coordinate part.

Finally, in [3] we presupposed the form of the solution, taking that the solution has to determine the odd coordinate part in terms of the even coordinate part. By the careful consideration of the σ-parity of the equations of motion, boundary and consistency conditions, and at the end by solving the simple equations for the unknown coefficients, we obtained the form of the solution.

The solutions obtained in these three ways are equivalent. The effective theory obtained on the solution, significantly differs from the flat background case. In the weakly curved background ($b_{\mu\nu} \neq 0$ and $B_{\mu\nu\rho} \neq 0$), the effective Lagrangian is defined on the doubled target space $(q^\mu, \tilde{q}^\mu)$. These effective coordinates appear naturally in the solution of the Neumann boundary conditions. The effective coordinates are dual to each other. The effective metric depends on the even effective coordinate q^μ, while the effective Kalb-Ramond field depends on the odd $\tilde{q}^\nu$. Because of this fact the term in the action containing the effective Kalb-Ramond field becomes Ω-even, which allows its survival. In the conventional space, with only one effective coordinate q^μ, the effective Kalb-Ramond field is eliminated, because it comes within the Ω-odd term in the action.

2. Open string theory in weakly curved background

The action describing the bosonic string moving in the background fields: metric $G_{\mu\nu}$ and Kalb-Ramond antisymmetric field $B_{\mu\nu}$ is given by

$$S = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x)\Big] \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{1}$$

($\varepsilon^{01} = -1$), where integration goes over two-dimensional world-sheet Σ parameterized by $\xi^0 = \tau$, $\xi^1 = \sigma$ with $\sigma \in [0, \pi]$. Here $x^\mu(\xi)$, $\mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, and we use the notation $\dot{x} = \frac{\partial x}{\partial \tau}$, $x' = \frac{\partial x}{\partial \sigma}$.

The background fields must satisfy the space-time equations of motion for the conformal invariance on the quantum level to be preserved. To the lowest order in slope parameter α', for the constant dilaton field $\Phi = const$ these equations have the form

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0\,, \quad D_\rho B^\rho{}_{\mu\nu} = 0, \tag{2}$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is a field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to space-time metric. We will consider the following particular solution of these equations, the *weakly curved* background

$$G_{\mu\nu} = const, \quad B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho = b_{\mu\nu} + h_{\mu\nu}(x), \tag{3}$$

where the parameter $b_{\mu\nu}$ is constant and $B_{\mu\nu\rho}$ is constant and infinitesimally small. Through the paper we will work up to the first order in $B_{\mu\nu\rho}$.

The minimal action principle for the open string leads to the equation of motion

$$\ddot{x}^\mu = x''^\mu - 2B^\mu{}_{\nu\rho}\dot{x}^\nu x'^\rho, \tag{4}$$

and the boundary conditions on the string endpoints. Choosing the Neumann boundary conditions we have

$$\gamma^0_\mu\Big|_{\sigma=0,\pi} = 0, \quad \gamma^0_\mu \equiv \frac{\delta\mathcal{L}}{\delta x'^\mu} = G_{\mu\nu}x'^\nu - 2B_{\mu\nu}\dot{x}^\nu. \tag{5}$$

We will solve these boundary conditions presupposing the form of the solution. We will discuss the effective theory obtained on the solution.

2.1. Solution of the boundary conditions

So, let us presuppose the form of the solution and analyse what such a form has to satisfy in order to actually be the solution. Defining the even and odd coordinate parts with respect to $\sigma = 0$

$$q^\mu(\sigma) = \frac{1}{2}\Big[x^\mu(\sigma) + x^\mu(-\sigma)\Big], \quad \bar{q}^\mu(\sigma) = \frac{1}{2}\Big[x^\mu(\sigma) - x^\mu(-\sigma)\Big], \tag{6}$$

we can separate the even and odd parts of the equation of motion (4) and the boundary condition (5) at $\sigma = 0$. The equations of motion become

$$\ddot{q}^\mu - q''^\mu = -2B^\mu{}_{\nu\rho}\Big[\dot{q}^\nu\bar{q}'^\rho + \dot{\bar{q}}^\nu q'^\rho\Big], \quad \ddot{\bar{q}}^\mu - \bar{q}''^\mu = -2B^\mu{}_{\nu\rho}\Big[\dot{q}^\nu q'^\rho + \dot{\bar{q}}^\nu\bar{q}'^\rho\Big], \tag{7}$$

and only the even part of γ^0_μ contributes to the boundary conditions at $\sigma = 0$

$$\gamma^0_\mu \equiv G_{\mu\nu}\bar{q}'^\nu - 2b_{\mu\nu}\dot{q}^\nu - 2h_{\mu\nu}(q)\dot{q}^\nu. \tag{8}$$

Let us suppose that the solution of the boundary conditions can be presented in the form where the first in τ and σ derivatives of the odd coordinate part will be expressed in terms of the first in τ and σ derivatives of the even coordinate part

$$\dot{\bar{q}}^\mu = -A^\mu_{1\nu}(\tilde{q})\dot{q}^\nu + 2\beta^\mu_{1\nu}(q)q'^\nu, \quad \bar{q}'^\mu = -A^\mu_{2\nu}(\tilde{q})q'^\nu + 2\beta^\mu_{2\nu}(q)\dot{q}^\nu. \tag{9}$$

We assume that the coefficient functions are linear. The characteristics of the coefficients A and β arguments, are dictated by the parity of both equations. The coefficients $A^\mu_{1\nu}$ and $A^\mu_{2\nu}$ are odd and as such they do not contain constant terms and they depend on some odd variable $\tilde{q}$, while $\beta^{\mu}_{1\nu}$ and $\beta^{\mu}_{2\nu}$ are even functions, depending on the (new independent) variable q^μ. Beside satisfying (8), the solution (9) must obey the equations of motion (7), the consistency condition $(\dot{\bar{q}}^\mu)' = (\bar{q}'^\mu)^{\cdot}$ and it must be in agreement with the zeroth order solution

$$\dot{\bar{q}}^\mu = 2b^\mu{}_\nu q'^\nu, \qquad \bar{q}'^\mu = 2b^\mu{}_\nu \dot{q}^\nu, \tag{10}$$

for $B_{\mu\nu\rho} = 0$.

Using the fact that the equations of motion (7) , the zeroth order solution (10) and the boundary conditions (8) are all invariant to the interchange of τ and σ derivatives, and that $A^\mu{}_\nu$ is infinitesimal, we can conclude that the ansatz must be invariant too and that $\beta^\mu{}_\nu(q) = (G^{-1})^{\mu\rho}B_{\rho\nu}(q)$. Now, substituting the redefined ansatz into the equations of motion (7), we obtain

$$\ddot{q}^\mu - q''^\mu = 12[h^\mu{}_\nu(\dot{q})b^\nu{}_\rho \dot{q}^\rho - h^\mu{}_\nu(q')b^\nu{}_\rho q'^\rho], \tag{11}$$

and

$$\dot{A}^\mu{}_\nu(\tilde{q})\dot{q}^\nu - A'^\mu_\nu(\tilde{q})q'^\nu + A^\mu{}_\nu(\tilde{q})(\ddot{q}^\nu - q''^\nu) = 2h'^\mu_\nu \dot{q}^\nu - 24h'^\mu_\nu(bq)(b\dot{q})^\nu, \tag{12}$$

while the consistency relation $(\dot{\bar{q}}^\mu)' = (\bar{q}'^\mu)^{\cdot}$ gives

$$2B^\mu{}_\nu(q)(\ddot{q}^\nu - q''^\nu) = \dot{A}^\mu{}_\nu(\tilde{q})q'^\nu - A'^\mu_\nu(\tilde{q})\dot{q}^\nu. \tag{13}$$

If

$$\dot{\tilde{q}}^\mu = q'^\mu, \qquad \tilde{q}'^\mu = \dot{q}^\mu. \tag{14}$$

these equations have the solution

$$A^\mu{}_\nu(q) = (G^{-1})^{\mu\rho}\Big[h(q) - 12bh(q)b - 12h(bq)b + 12bh(bq)\Big]_{\rho\nu}, \tag{15}$$

with the property $(GA)_{\mu\nu} = -(GA)_{\nu\mu}$. Finally, we can write the space-time coordinates satisfying the boundary condition at $\sigma = 0$ as

$$\begin{aligned} \dot{x}^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]\dot{q}^\nu + 2[G^{-1}B(q)]^\mu{}_\nu q'^\nu, \\ x'^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})]q'^\nu + 2[G^{-1}B(q)]^\mu{}_\nu \dot{q}^\nu. \end{aligned} \tag{16}$$

The same expressions have been obtained in ref. [1], using canonical methods.

instead of (6), we define even and odd variables with respect to $\sigma = \pi$

$${}^\star q^\mu(\sigma) = \frac{1}{2}\Big[x^\mu(\pi+\sigma) + x^\mu(\pi-\sigma)\Big], \quad {}^\star\bar{q}^\mu(\sigma) = -\frac{1}{2}\Big[x^\mu(\pi+\sigma) - x^\mu(\pi-\sigma)\Big]. \tag{17}$$

Applying the analogous procedure as for the case $\sigma = 0$, we obtain

$$\dot{x}^\mu(\sigma) = \Big[\delta^\mu_\nu - A^\mu{}_\nu[{}^\star\tilde{q}(\pi-\sigma)]\Big]{}^\star\dot{q}^\nu(\pi-\sigma) + 2\Big[G^{-1}B[{}^\star q(\pi-\sigma)]\Big]^\mu{}_\nu{}^\star q'^\nu(\pi-\sigma),$$
$$x'^\mu(\sigma) = \Big[\delta^\mu_\nu - A^\mu{}_\nu[{}^\star\tilde{q}(\pi-\sigma)]\Big]{}^\star q'^\nu(\pi-\sigma) + 2\Big[G^{-1}B[{}^\star q(\pi-\sigma)]\Big]^\mu{}_\nu{}^\star\dot{q}^\nu(\pi-\sigma). \tag{18}$$

Note that if

$$q^\mu(\sigma) = {}^\star q^\mu(\pi-\sigma), \qquad \bar{q}^\mu(\sigma) = {}^\star\bar{q}^\mu(\pi-\sigma), \tag{19}$$

then the solutions (16) and (18) coincide, and from the relation (19) follows the 2π-periodicity of x^μ. So, if we extend the σ domain and demand $2\pi-$periodicity of the original variable $x^\mu(\sigma+2\pi) = x^\mu(\sigma)$, the relation (16) solves both constraints at $\sigma = 0$ and $\sigma = \pi$.

Let us stress that the solution of the boundary condition does not depend on one effective variable only, but on the two variables q^μ and $\tilde{q}^\mu$, connected by the relation (14). So, we obtained some non-geometric space with the doubled number of degrees of freedom but with the constraint (14).

2.2. Effective theory

Substituting the solution (16) into the Lagrangian (1) we obtain the effective Lagrangian. Because our basic variable $q^\mu(\sigma)$ contains only even powers of σ, it is convenient to regard it as the even function $q^\mu(-\sigma) = q^\mu(\sigma)$ on the interval $\sigma \in [-\pi, \pi]$. Hereafter, we will consider the action $S^{eff} = \int d\tau \int_{-\pi}^{\pi} d\sigma \mathcal{L}^{eff}$, and consequently, the terms of the effective metric which depend on $\tilde{q}$ and the term of effective Kalb-Ramond field which depends on q will disappear, so that

$$S^{eff} = \kappa \int_{\Sigma_1} d^2\xi \Big[\frac{1}{2}\eta^{\alpha\beta} G^{eff}_{\mu\nu}(q) + \epsilon^{\alpha\beta} B^{eff}_{\mu\nu}(2b\tilde{q})\Big]\partial_\alpha q^\mu \partial_\beta q^\nu. \tag{20}$$

Here Σ_1 marks the changed sigma domain $\sigma \in [-\pi, \pi]$. The effective background fields are equal to

$$G^{eff}_{\mu\nu}(q) = G^E_{\mu\nu}(q), \quad B^{eff}_{\mu\nu}(2b\tilde{q}) = -\frac{\kappa}{2}[g\Delta\theta(2b\tilde{q})g]_{\mu\nu}, \tag{21}$$

where

$$G^E_{\mu\nu}(x) \equiv G_{\mu\nu} - 4B_{\mu\rho}(x)(G^{-1})^{\rho\sigma}B_{\sigma\nu}(x), \tag{22}$$

is the open string metric, and $\Delta\theta$ is the infinitesimal part of the so called non-commutativity parameter

$$\theta^{\mu\nu} = -\frac{2}{\kappa}\Big[G_E^{-1}BG^{-1}\Big]^{\mu\nu} = \theta_0^{\mu\nu} - \frac{2}{\kappa}\Big[g^{-1}(h + 4bhb)g^{-1}\Big]^{\mu\nu}. \tag{23}$$

It is defined in analogy with that of the flat space-time introduced in [4]. The constant parts of the effective metric and the non-commutativity parameter are denoted by $g_{\mu\nu} = G^E_{\mu\nu}(0)$ and $\theta_0^{\mu\nu} = \theta^{\mu\nu}(0) = -\frac{2}{\kappa}\left[g^{-1}bG^{-1}\right]^{\mu\nu}$.

So, the complete transition from the original theory (1) to the effective theory (20) consists of *the transition from conventional to the doubled geometry*

$$x^\mu \to q^\mu,\, \tilde{q}^\mu \tag{24}$$

and *the background field transition*

$$G_{\mu\nu} \to G^{eff}_{\mu\nu}(q), \qquad B_{\mu\nu}(x) \to B^{eff}_{\mu\nu}(2b\tilde{q})\,. \tag{25}$$

The first transition says that while the original theory is defined on the geometric target space, the effective one is defined on the enlarged, the so called doubled target space, given in terms of both the effective coordinate q^μ and its T-dual $\tilde{q}^\mu$ (explained in the next section). Such a space is non-geometrical space [8]. In our case of the weakly curved background, the non-geometrical space arose naturally in the solution of the Neumann boundary conditions.

The appearance of the doubled target space allowed the string to see the effective background field $B^{eff}_{\mu\nu}$. In fact, the effective theory is Ω-even projection of the initial one, therefore in the geometric background the term with Kalb-Ramond field vanished as the Ω-odd term in the action. But, in doubled target space Kalb-Ramond field depends on the T-dual coordinate $\tilde{q}^\mu$, which is Ω-odd. So, the corresponding term in the action becomes Ω-even and can not be projected out by the world-sheet parity projection.

3. Doubled geometry of the effective theory

The effective theory for the constant background is the theory of the unoriented closed string. It is well known that it does not contain the Kalb-Ramond field. The explanation of Ref.[5], is that Kalb-Ramond field appears within the term $B_{\mu\nu}\dot{q}^\mu q'^\nu$, which is odd under σ-parity and consequently does not contribute to the action.

For the weakly curved background, there are two somewhat unexpected things in the effective theory (20). Not only that the non-trivial Kalb-Ramond field $B^{eff}_{\mu\nu}$ appears, but it is coordinate dependent. Moreover, it does not depend on the coordinate q^μ but on $\tilde{q}^\mu$. Let us analyze and explain these results.

In the case of the weakly curved background the effective Kalb-Ramond field $B^{eff}_{\mu\nu}(2b\tilde{q})$ is proportional to $\tilde{q}^\mu$, and consequently it is odd under σ-parity transformation

$$\Omega B^{eff}_{\mu\nu}[2b\tilde{q}(\sigma)] = -B^{eff}_{\mu\nu}[2b\tilde{q}(\sigma)].$$

This makes the term $B^{eff}_{\mu\nu}(2b\tilde{q})\dot{q}^\mu q'^\nu$ Ω-even and allows its survival.

The relations in (14) help us to find the interpretation of $\tilde{q}^\mu$. Note that it is enough to consider $\tilde{q}^\mu$ in the zeroth order, because it appears as an argument of $B^{eff}_{\mu\nu}$ only, which is the infinitesimal of the first order. The solution of the zeroth order equation of motion $\partial_+\partial_- q^\mu = 0$ for Ω-even variable q^μ has a form

$$q^\mu(\sigma) = f^\mu(\sigma^+) + f^\mu(\sigma^-). \quad (\sigma^\pm = \tau \pm \sigma) \tag{26}$$

With the help of relation $\dot{q}^\mu(\sigma) = f'^\mu(\sigma^+) - f'^\mu(\sigma^-)$, both equations from (14) produce

$$\tilde{q}^\mu(\tau,\sigma) = f^\mu(\sigma^+) - f^\mu(\sigma^-) + f_0. \tag{27}$$

The integration constant f_0 in (27) is zero because the odd variable $\tilde{q}^\mu$ can not contain the constant part. This means that $\tilde{q}^\mu(\tau,\sigma)$ is T-dual mapping of the effective coordinate $q^\mu(\tau,\sigma)$ (see for example (17.76) of Ref. [5] or eq. (6.17) of Ref.[6]).

Let us comment the duality between q and $\tilde{q}$ in general case. For the weakly curved background Ref. [7], the dual coordinate y_μ can be expressed in terms of the original one x^μ as

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}[x]\partial_\pm x^\nu \mp 2\beta^\mp_\mu[x], \tag{28}$$

where $\Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ and $\beta^\alpha_\mu[x] \equiv \partial_\mu B_{\nu\rho}\epsilon^{\alpha\beta}x^\nu\partial_\beta x^\rho$ is infinitesimal. In the present article, the role of the initial coordinate x^μ takes q^μ, and the role of the dual coordinate y_μ takes $\tilde{q}^\nu$. Projecting (28) into odd and even part and neglecting first order in $B_{\mu\nu\rho}$ terms, one obtains

$$\dot{\tilde{q}}^\mu \cong q'^\mu, \quad \tilde{q}'^\mu \cong \dot{q}^\mu. \tag{29}$$

These are just the expressions (14). So, q^μ and $\tilde{q}^\nu$ are not just duals in the zeroth order but truly the dual coordinates.

So, in the effective theory, the effective metric depends on the effective coordinate q^μ and the effective Kalb-Ramond field on its T-dual $\tilde{q}^\mu$. This kind of background is seen to be possible in the so called doubled formulation (the analysis of the non-geometrical backgrounds where the T-duality is allowed as the transition function [8]).

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