New Symmetries and Particular Solutions for 2D Black-Scholes Model*

R. Cimpoiasu†
University of Craiova, 13 A.I.Cuza, 200585 Craiova, ROMANIA

R. Constantinescu‡
University of Craiova, 13 A.I.Cuza, 200585 Craiova, ROMANIA

Abstract

Starting from a general second order differential equation, the Lie symmetry analysis of the $2D$ Black-Scholes model for option pricing is performed. The corresponding Lie algebra is identified and the algorithm for constructing exact (invariant) solutions under one-dimensional and two-dimensional subalgebras of this Lie algebra is illustrated. By applying the inverse symmetry problem, more general $2D$ models equivalent, as symmetry group, with Black-Scholes or Jacobs-Jones ones, are pointed out.

PACS: 05.45.-a; 89.65.Gh; 02.20.Tw
Keywords: option pricing models, symmetry analysis.
MSC2000: 70G65, 91B26.

1. Introduction

The description of social and of individual wealth is an important issue, as well for economists as for mathematicians and physicists. Fields of study as financial mathematics, econophysics or even sociophysics [1] refer to this issue and share the theory of stochastic processes as common "golden thread". Many mathematical models have been proposed on this basis, in order to describe the evolution of the value of financial derivatives. Financial models were generally formulated in terms of stochastic differential equations. However, soon it was obvious that under certain restrictions these models could be written as linear evolutionary partial differential equations (PDEs) with variable coefficients.

* Work supported by the Romanian Ministry of Education, Research and Innovation, through the National Council for Scientific Research in Higher Education (CNCSIS), in the frame of the Programme "Ideas", grant code ID 418/2008.

† e-mail address: rodicimp@yahoo.com
‡ e-mail address: rconsta@yahoo.com

The cardinal contribution to the option pricing theory came from Fischer Black and Myron Scholes through the option pricing formula [2, 3] ,the one for which they won the Nobel Prize for Economics in 1997, together with Robert Merton. Another interesting transfer from Physics to Finance is due to Dragulescu and Yakovenko who proposed many models of money, wealth and income distributions starting from the claim that the probability distribution follows the Boltzmann-Gibbs law [4].

In this paper we intend to apply the symmetry method to a generalized form of the $2D$ BS model. Symmetry-related methods [5, 6, 7] play an important role in finding exact solutions of PDE's and in checking their geometrical properties. An important application of Lie theory of symmetry groups for PDE's consists in obtaining group invariant solutions. Its importance lies within the fact that it usually describes asymptotic behavior or display the structure for the singularities of a general solution.

In recent years, a number of studies applied the Lie symmetry method to equations coming from finance, as in [8, 9]. In this paper we deal with European call options on a basket of two assets x, y with mean tendencies (or expected rates of returns) μ_i, $i = 1, 2$, volatilities σ_i and correlation ρ. We assume that x, y are governed by stochastic processes of the form:

$$\begin{aligned} dx &= \mu_1 x dt + \sigma_1 x dW^1 \\ dy &= \mu_2 y dt + \sigma_2 y dW^2 \\ \rho &= d(W^1, W^2) \end{aligned} \tag{1}$$

The financial parameters μ_i, σ_i and ρ are arbitrary constants and dW^1, dW^2 denote increments in the standard Wiener process [10]. An option u with pay-off $u_T(x, y)$ at maturity T will satisfy a two-dimensional Black-Scholes partial differential equation in $\mathcal{R}_+^2 \times [0, T]$:

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}\sigma_1^2 x^2 u_{2x} + \frac{1}{2}\sigma_2^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xy u_{xy} - ku = 0$$
$$u(x, y, T) = u_T(x, y) \tag{2}$$

with k an arbitrary constant.

The present paper will be centered on this equation and on a generalized version of it. After this introduction, in Section 2 of the paper the Lie symmetry analysis of (2) will be presented. We shall compute the symmetry generators and we shall identify the corresponding Lie algebra for this model, starting from the determining system for a more general second order PDE presented in [11]. The next section of the paper will deal with the problem of finding invariant solutions of $2D$ BS equation in respect with various subalgebra of the whole Lie algebra. We shall illustrate the algorithm for constructing such solutions and we shall effectively obtain new solutions, invariant under transformations generated by some one-dimensional and two-dimensional subalgebras. The inverse symmetry problem is proposed in Section 4. We obtain the most general class of dynamical equations, which are equivalent, from the point of view of their symmetry group. The more complex Jacobs-Jones financial model with arbitrary parameters [12], may be identified within this general class. By applying again the direct symmetry approach to the master differential system, we shall discover more symmetry operators for the Jacobs-Jones model. They correspond to particular values or relations among the parameters. The paper will end by some concluding remarks.

2. Symmetry analysis for the 2D Black-Scholes model

Point-like Lie symmetries admitted by the $1D$ Black-Scholes equation or by associated potential differential system are performed, for example in [13] and respectively in [14]. In this section we are going to compute Lie symmetry generators and their associated algebra for the $2D$ model (2), starting from already known results.

2.1. Determining system for Lie symmetries

In our previous work [11] we studied the Lie symmetry problem for a $2D$ dynamical system described by a second order partial differential equation. In this paper we shall consider a particular case, with an equation of the form:

$$\begin{aligned} u_t \;=\;& A(x,y)u_{xy} + B(x,y)u_x u_y + C(x,y)u_{2x} + D(x,y)u_{2y} + \\ & E(x,y)u_y + F(x,y)u_x + G(x,y,u) \end{aligned} \tag{3}$$

The general infinitesimal symmetry operator is considered of the general form:

$$\widetilde{X} = \varphi(x,y,t,u)\frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u} \tag{4}$$

With no generality loss, we can consider $\varphi = c_0$. With this choice, the determining system which links the coefficients functions $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$, $F(x,y)$, $G(x,y,u)$ to infinitesimal coefficients $\xi(x,y,t,u)$, $\eta(x,y,t,u)$, $\phi(x,y,t,u)$ has the form:

$$\begin{aligned} & B\xi_y - C\phi_{2u} = 0, \\ & B\eta_x - D\phi_{2u} = 0, \\ & A\eta_y - \eta A_y + A\xi_x - \xi A_x + 2D\xi_y + 2C\eta_x = 0, \\ & A\xi_y + 2C\xi_x - \xi C_x - \eta C_y = 0, \\ & A\eta_x + 2D\eta_y - \eta D_y - \xi D_x = 0, \\ & -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_x\xi - B_y\eta = 0, \\ & -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_x\xi - E_y\eta + A\eta_{xy} - A\phi_{xu} + \\ & \qquad\qquad C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu} = 0, \end{aligned} \tag{5}$$

$$
\begin{aligned}
&-\xi_t - B\phi_y + F\xi_x + E\xi_y - F_x\xi - F_y\eta + A\xi_{xy} - A\phi_{yu} + \\
&\qquad\qquad\qquad\qquad C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu} = 0, \\
&\phi_t + G\phi_u - F\phi_x - E\phi_y - G_x\xi - G_y\eta - \\
&\qquad\qquad\qquad\qquad -G_u\phi - A\phi_{xy} - C\phi_{2x} - D\phi_{2y} = 0.
\end{aligned}
$$

In order to solve this system for the $2D$ Black-Scholes equation (2), we have to consider:

$$
\begin{aligned}
A &= -\rho\sigma_1\sigma_2 xy;\ B = 0;\ C = -\frac{1}{2}\sigma_1^2 x^2;\ D = -\frac{1}{2}\sigma_2^2 y^2; \qquad (6)\\
E &= -\mu_2 y;\ F = -\mu_1 x; G = ku \qquad (7)
\end{aligned}
$$

We may also choose $c_0 = 1$. Using a computational algorithm (Maple program) to solve the system (5), we come to the general solution:

$$
\xi = \frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 \ln x - \sigma_1 \ln y] + x(c_1 t + c_2) \tag{8}
$$

$$
\eta = \frac{-c_3 y}{\rho\sigma_1}[\rho\sigma_1 \ln y - \sigma_2 \ln x] + y(c_4 t + c_5) \tag{9}
$$

$$
\phi = \omega + \beta u \tag{10}
$$

where $\{c_i,\ i = 1, ..., 6\}$are arbitrary constants, ω is any solution of (2), and β has the more complicated expression below:

$$
\begin{aligned}
\beta \ =\ & \frac{1}{\rho\,(\sigma_1)^2\,(\sigma_2)^2\,(1-\rho^2)} \\
& \times \left\{ \left[-c_1 \frac{\rho^2\sigma_1\sigma_2}{2} + c_3 \frac{\sigma_1\sigma_2}{2}\left[(\sigma_1)^2 - 2\mu_1\right](\rho^2 - 1) + c_4\rho\,(\sigma_1)^2 \right] \ln y + \right. \\
& + \left. \left[c_1\rho\,(\sigma_2)^2 - c_3 \frac{\sigma_1\sigma_2}{2}\left[(\sigma_2)^2 - 2\mu_2\right](\rho^2 - 1) - c_4\rho^2\sigma_1\sigma_2 \right] \ln x + \gamma \right\}
\end{aligned}
$$

with:

$$
\begin{aligned}
\gamma \ =\ & t \left\{ c_1 \left[\frac{\rho\,(\sigma_2)^2}{2}\left[(\sigma_1)^2 - 2\mu_1\right] - \frac{\rho^2\sigma_1\sigma_2}{2}\left[(\sigma_2)^2 - 2\mu_2\right] \right] \right. \\
& \left. + c_4 \left[\frac{\rho\,(\sigma_1)^2}{2}\left[(\sigma_2)^2 - 2\mu_2\right] - \frac{\rho^2\sigma_1\sigma_2}{2}\left[(\sigma_1)^2 - 2\mu_1\right] \right] \right\} \\
& + c_6\rho\,(\sigma_1)^2\,(\sigma_2)^2\,(1-\rho^2)
\end{aligned}
$$

2.2. Algebra of the Lie operators

According to (10), we may decompose the Lie operator (4) in the form $\widetilde{X} = X + X_\omega$, where:

$$X = \frac{\partial}{\partial t} + \left\{ \frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 \ln x - \sigma_1 \ln y] + x(c_1 t + c_2) \right\} \frac{\partial}{\partial x}$$
$$+ \left\{ \frac{-c_3 y}{\rho\sigma_1}[\rho\sigma_1 \ln y - \sigma_2 \ln x] + y(c_4 t + c_5) \right\} \frac{\partial}{\partial y} + \beta(x,y,t)u\frac{\partial}{\partial u} \qquad (11)$$

$$X_\omega = \omega \frac{\partial}{\partial u} \qquad (12)$$

Let $\widetilde{\Lambda}$ be the set of all Lie operators (4) which satisfy the differential system (5). It possesses a natural Lie algebra structure and it may be decomposed into a direct sum:

$$\widetilde{\Lambda} = \Lambda + \Lambda_\omega = \{X\} \oplus \{X_\omega\} \qquad (13)$$

where Λ consists in all the operators from $\widetilde{\Lambda}$ with $\omega = 0$ and Λ_ω is the set of all the operators from $\widetilde{\Lambda}$ with $c_1 = c_2 = ... = c_6 = 0$.More precisely,

$$\Lambda_\omega = \left\{ \omega(t,x,y)\frac{\partial}{\partial u} \mid \omega \text{ solution of } (2) \right\} \qquad (14)$$

It is an abelian and infinite dimensional Lie algebra, its subsequent linear space being isomorphic to the space of all solutions of (2).

As a conclusion, the solution of the determining equations (5) provides the infinite dimensional vector space of the infinitesimal symmetries for equation (2), spanned by the following operators:

$$X_1 = \frac{\partial}{\partial t}$$
$$X_2 = xt\frac{\partial}{\partial x} + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)} \Big\{ (\sigma_2)^2 \ln x - \rho\sigma_1\sigma_2 \ln y +$$
$$\frac{t}{2}\left[(\sigma_2)^2 \left[(\sigma_1)^2 - 2\mu_1 \right] - \rho\sigma_1\sigma_2 \left[(\sigma_2)^2 - 2\mu_2 \right] \right] \Big\} u\frac{\partial}{\partial u}$$
$$X_3 = x\frac{\partial}{\partial x}$$
$$X_4 = x\left(-\frac{\sigma_1}{\rho\sigma_2} x \ln y + x \ln x \right)\frac{\partial}{\partial x} + \left(\frac{\sigma_2}{\rho\sigma_1} y \ln x - y \ln y \right)\frac{\partial}{\partial y} +$$
$$\left\{ \frac{\left[2\mu_1 - (\sigma_1)^2 \right] \ln y - \left[2\mu_2 - (\sigma_2)^2 \right] \ln x}{2\rho\sigma_1\sigma_2} u \right\} \frac{\partial}{\partial u} \qquad (15)$$

$$X_5 = yt\frac{\partial}{\partial y} + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_1)^2\ln y - \rho\sigma_1\sigma_2\ln x + \frac{t}{2}\Big[(\sigma_1)^2\Big[(\sigma_2)^2 - 2\mu_2\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_1)^2 - 2\mu_1\Big]\Big]\Big\}u\frac{\partial}{\partial u}$$
$$X_6 = y\frac{\partial}{\partial y}\,, \qquad X_7 = u\frac{\partial}{\partial u}\,, \qquad X_\omega = \omega(t,x,y)\frac{\partial}{\partial u}\,.$$

When the finite-dimensional Lie Algebra generated by $\Lambda = \{X_1, X_2, ..., X_7\}$ is computed, the following non-vanishing relations are obtained:

$$\begin{aligned}
&[X_1, X_2] = X_3 + (k_3 + k_4)X_7,\ [X_1, X_5] = X_6 + (k_6 + k_7)X_7,\\
&[X_2, X_3] = -k_1X_7,\ [X_2, X_4] = X_2 + k_9X_5,\ [X_2, X_6] = -k_2X_7,\\
&[X_3, X_4] = X_3 + k_9X_6 + k_{10}X_7,\ [X_3, X_5] = k_2X_7,\\
&[X_4, X_5] = X_5 - k_8X_2,\ [X_4, X_6] = -k_8X_3 + X_6 - k_{11}X_7,\\
&[X_5, X_6] = -k_5X_7
\end{aligned} \tag{16}$$

where we introduced arbitrary constants k_i, $i = \overline{1,11}$ with the following expressions:

$$\begin{aligned}
k_1 &= \frac{1}{(\sigma_1)^2(1-\rho^2)},\ k_2 = \frac{-\rho}{\sigma_1\sigma_2(1-\rho^2)},\ k_3 = \frac{k_1\Big[(\sigma_1)^2 - 2\mu_1\Big]}{2},\\
k_4 &= \frac{k_2\Big[(\sigma_2)^2 - 2\mu_2\Big]}{2},\ k_5 = \frac{1}{(\sigma_2)^2(1-\rho^2)}\ ,\ k_6 = \frac{k_4k_5}{k_2}
\end{aligned} \tag{17}$$

$$k_7 = \frac{k_2k_3}{k_1},\ k_8 = -\frac{\sigma_1}{\rho\sigma_2},\ k_9 = -\frac{k_1}{k_2},\ k_{10} = \frac{k_4}{k_2}\frac{1}{\rho\sigma_1\sigma_2},\ k_{11} = -\frac{k_3}{k_1}\frac{1}{\rho\sigma_1\sigma_2}$$

In addition, when we take into consideration the generator X_ω, we obtain the following non-vanishing relations:

$$\begin{aligned}
&[X_1, X_\omega] = X_{\omega_t}, \qquad [X_3, X_\omega] = X_{x\omega_x},\\
&[X_2, X_\omega] = X_{xt\omega_x - \omega[k_1\ln x + k_2\ln y + t(k_3+k_4)]},\\
&[X_4, X_\omega] = X_{\omega_x(k_8x\ln y + x\ln x) + \omega_x(k_9y\ln x - y\ln y) - \omega(k_{10}\ln x + k_{11}\ln y)},\\
&[X_5, X_\omega] = X_{yt\omega_y - \omega[k_5\ln y + k_2\ln x + t(k_6+k_7)]},\\
&[X_6, X_\omega] = X_{y\omega_y}, \qquad [X_7, X_\omega] = X_{-\omega}
\end{aligned} \tag{18}$$

3. Invariant solutions for the 2D Black-Scholes model

As we already mentioned, Lie (classical) symmetries of an arbitrary dynamical systems are useful for finding invariant solutions. An invariant solution in relation to a given subgroup of the symmetry group is a solution which is unalterable under the action of the subgroup transformations. Invariant solutions can be expressed via subgroup invariants [15].

We shall illustrate the calculation of invariant solutions for the $2D$ Black-Scholes equation by considering two cases of subalgebras of the general algebra (13). More precisely, we shall investigate now the $1D$ and, respectively, $2D$ subalgebras.

3.1. Invariant solutions generated by 1D subalgebras

As one-parameter subgroups which may be generated by the operators (15) we will consider the cases: (i) $X^{(1)} = X_1 + X_2 + X_4$; (ii) X_2 or similarly X_5; (iii) X_4. Let us mention that operators X_7 and X_ω do not provide invariant solutions and that the same procedure can be applied to whatever linear combination (with constant coefficients) of the basic symmetry generators (15).

(i) Let us start by the generator:

$$X^{(1)} = \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \tag{19}$$

The functionally independent invariants of this subgroup are found by integrating the characteristic equations:

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dy}{0} = \frac{du}{u} \tag{20}$$

and are given by the expressions:

$$I_1 = y,\ I_2 = t - \ln x,\ I_3 = \frac{u}{x} \tag{21}$$

The invariant solution may be expressed in the form $I_3 = \Psi^{(1)}(I_1, I_2)$, or

$$u^{(1)} = x\ \Psi^{(1)}(y, z),\ \text{where } z = t - \ln x \tag{22}$$

Substituting into equation (2) we obtain, for $\Psi^{(1)}(y, z)$, the partial differential equation of second order:

$$\frac{(\sigma_1)^2}{2}\Psi^{(1)}_{2z} + \frac{(\sigma_2)^2}{2}y^2\Psi^{(1)}_{2y} - \rho\sigma_1\sigma_2 y\Psi^{(1)}_{zy} + \left[1 - \mu_1 - \frac{(\sigma_1)^2}{2}\right]\Psi^{(1)}_z + \\ (\mu_2 + \rho\sigma_1\sigma_2)y\Psi^{(1)}_y + (\mu_1 - k)\Psi^{(1)} = 0 \tag{23}$$

For sake of simplicity, we will consider the solution of the previous equation for the following particular values of parameters: $\sigma_1 = \sigma_2 = k = 1/2$, $\mu_1 = \mu_2 = 1/10$, $\rho = 3/5$. In these circumstances, the previous equation admits a solution of the form:

$$\Psi^{(1)}(y,z) = c_1 + c_2 y^{(-1)} + c_3 e^{(-31/5)z} \tag{24}$$

Consequently, the invariant solution (22), for original variables, becomes:

$$u^{(1)}(t,x,y) = x[c_1 + c_2 y^{(-1)} + c_3 e^{(-31/5)t} x^{(31/5)}] \tag{25}$$

(ii) Let us derive now invariant solutions generated by the operator X_2. We rely this time on the assertion that the function $u^{(2)} = \Psi^{(2)}(t,x,y)$ is a group invariant solution of (2) if:

$$X_2[u^{(2)} - \Psi^{(2)}(t,x,y)]\,|_{u^{(2)}=\Psi^{(2)}} = 0 \tag{26}$$

This condition is equivalent to the partial differential equation:

$$-tx\Psi_x^{(2)} + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_2)^2\ln x - \rho\sigma_1\sigma_2\ln y +$$
$$\frac{t}{2}\Big[(\sigma_2)^2\Big[(\sigma_1)^2 - 2\mu_1\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_2)^2 - 2\mu_2\Big]\Big]\Big\}\Psi^{(2)} = 0 \tag{27}$$

In order to simplify the expression of the solution associated to (27), we choose again particular values for the parameters:

$$\rho = 3/5 \in (0,1);\ \sigma_1 = \sigma_2 = 1/2;\ \mu_1 = \mu_2 = 1/10 \tag{28}$$

With this choice, the solution of (27) has the form:

$$\Psi^{(2)} = \chi(y,t)x^{\left(\frac{1}{16}\right)}y^{\left(\frac{-15}{4t}\ln x\right)}e^{\left(\frac{25}{8t}(\ln x)^2\right)} \tag{29}$$

with $\chi(y,t)$ an arbitrary function. By substituting the solution (29) into equation (2) with the values (28) of the parameters, we obtain a partial differential equation for $\chi(y,t)$:

$$t^2\left[\chi_{2y}^2 + 8\chi_t + \frac{7}{8}y\chi_y - (8k + \frac{11}{128})\right]\chi$$
$$+\frac{t}{2}\left[\frac{9}{16}(\ln y)\chi - 9y(\ln y)\chi_y + \frac{7}{2}\chi\right] + \frac{225}{16}(\ln y)^2\chi = 0. \tag{30}$$

Let us choose to solve the previous equation by considering alternatively either y or t as parameters. In the first situation (y as parameter and $\ln y \equiv \gamma$), the solution becomes:

$$\chi(t;\gamma) = t^{\frac{-1}{32}\left(7+\frac{9\gamma}{8}\right)}e^{\left(\frac{1024kt^2+11t^2+1800\gamma^2}{1024t}\right)} \tag{31}$$

Consequently, under the previous condition, the invariant solution generated by the symmetry operator X_2 has the final expression:

$$u^{(2)}(t,x;\gamma) = x^{\frac{1}{4}\left(\frac{1}{4}-\frac{15\gamma}{t}+\frac{25\ln x}{2t}\right)} t^{\frac{-1}{32}\left(7+\frac{9\gamma}{8}\right)} e^{\left[\frac{t^2(1024k+11)+1800\gamma^2}{1024t}\right]} \tag{32}$$

In the second case, when $t \equiv n$ is considered as parameter, the equation (30) can be reduced to a Riccati equation, by using the substitution $\frac{f\,'(y)}{f(y)} \equiv R(y)$. It has the form:

$$R'(y) + R^2(y) + \frac{\left[\frac{7}{8} - \frac{9}{2n}\ln y\right]}{y} R(y) + \frac{\left[-\left(8k+\frac{11}{128}\right) + \frac{9}{32n}\ln y + \frac{7}{4n} + \frac{225}{16n^2}(\ln y)^2\right]}{y^2} = 0 \tag{33}$$

The previous equation admits a solution expressed in terms of WhittakerW and WhittakerM special functions. We have to note that, due to the symmetry of equation (2) in the variables x and y, similar results could be found if we would consider the $1D$ subalgebra generated by X_5.

(iii) The last case we are considering corresponds to invariant solutions generated by X_4. The function $u^{(3)} = \Psi^{(3)}(t,x,y)$ is a group invariant solution of (2) provided that:

$$X_4[u^{(3)} - \Psi^{(3)}(t,x,y)]\,|_{u^{(3)}=\Psi^{(3)}} = 0 \tag{34}$$

This condition, under particular values (28) of parameters, is equivalent to the following partial differential equation for $\Psi^{(3)}(t,x,y)$:

$$(\ln x - \ln y)\Psi^{(3)} + 4x\ln x\Psi^{(3)}_x - 4y\ln y\Psi^{(3)}_y = 0 \tag{35}$$

This equation admits the solution:

$$\Psi^{(3)}(t,x,y) = \Omega\left[t, -\frac{8}{5}(\ln x)^2 - \frac{1}{10}\left(\ln\left(\frac{y^5}{x^3}\right)\right)^2\right] e^{P(x)} \tag{36}$$

where $P(x)$ has an integral form.

By substituting (36) in (2) and observing (28), we could obtain in principle the solution of equation (2) for its original variables.

3.2. Invariant solutions generated by 2D subalgebras

Let us consider now three examples of invariant solutions corresponding to two-dimensional subalgebras of the Lie algebra (15). We will see that these examples generate solutions of equation (2) by reducing the problem to the one of solving a linear second order Riccati equation.

(i) The subalgebra generated by $\{X_1, X_6\}$ has the independent invariants $I_1 = x$, $I_2 = u$. The corresponding invariant solution has the form

$$I_2 = \Phi(I_1) \ \Leftrightarrow \ u = \Phi(x) \tag{37}$$

By substituting (37) into equation (2), we obtain the following ordinary differential equation for the function Φ :

$$\frac{(\sigma_1)^2}{2} x^2 \Phi'' + \mu_1 x \Phi' - k\Phi = 0 \tag{38}$$

Using the standard substitution $H = \frac{\Phi'}{\Phi}$, we reduce it to the Riccati equation:

$$H' + H^2 + \frac{2\mu_1}{(\sigma_1)^2 x} H - \frac{2k}{(\sigma_1)^2 x^2} = 0 \tag{39}$$

which admits the solution:

$$\begin{aligned} H(x) &= \frac{1}{2(\sigma_1)^2 x} \left\{ (\sigma_1)^2 - 2\mu_1 - \sqrt{8k(\sigma_1)^2 + \left[(\sigma_1)^2 - 2\mu_1\right]^2} \right\} \\ & \left\{ \tanh \left[\frac{\sqrt{8k(\sigma_1)^2 + \left[(\sigma_1)^2 - 2\mu_1\right]^2} (d_1 - \ln x)}{2(\sigma_1)^2} \right] \right\} \end{aligned} \tag{40}$$

with d_1 arbitrary constant.

(ii) The subalgebra $\{X_1 + X_7, X_6\}$ has the independent invariants $I_1 = x$, $I_2 = ue^{-t}$. Thus, the invariant solution has the form:

$$u = e^t \Phi(x) \tag{41}$$

The substitution into the master equation (2) brings us to:

$$\frac{(\sigma_1)^2}{2} x^2 \Phi'' + \mu_1 x \Phi' + (1-k)\Phi = 0 \tag{42}$$

Similarly with the previous example, (42) reduces itself to the Riccati equation (39) where k is substituted by $k-1$.

(iii) The subalgebra $\{X_1,\ X_6 + X_7\}$ has the independent invariants $I_1 = x$, $I_2 = \frac{u}{y}$. The invariant solution has the form:

$$u = y\ \Phi(x) \tag{43}$$

with $\Phi(x)$ solution of the equation:

$$\frac{(\sigma_1)^2}{2} x^2 \Phi'' + (\mu_1 + \rho\sigma_1\sigma_2) x \Phi' + (\mu_2 - k)\Phi = 0 \tag{44}$$

With the substitutions $\mu_1 \longrightarrow (\mu_1 + \rho\sigma_1\sigma_2)$ and $k \longrightarrow (k - \mu_2)$, this last equation reduces itself again to a Riccati equation, similar with (39).
Remark 1: The operator X_ω from (15) does not provide invariant solutions through the direct method. However, we can use this operator to generate new solutions, using the fact that the infinite set of operators of the form X_ω is an ideal of the whole Lie algebra:

$$[X_\omega, X] = X_{\overline{\omega}} \tag{45}$$

From a mathematical point of view, it means that, if $u = \omega(t, x, y)$ is a solution of equation (2) and X is whatever Lie symmetry operator from (15), $\overline{\omega}(t, x, y)$ is a new solution of (2). Let us verify this assertion by choosing $X = X_5$ from (15) and, as initial solution ω, the solutions given in the previous three examples.

For solution (37), $\omega(t, x, y) = \Phi(x)$, where $\Phi(x)$ is defined by differential equation (38). Then, formula (45) becomes:

$$\begin{aligned}[X_\omega, X_5] \equiv \overline{\omega_1}\frac{\partial}{\partial u} = \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_1)^2\ln y - \rho\sigma_1\sigma_2\ln x + \\ \frac{t}{2}\Big[(\sigma_1)^2\Big[(\sigma_2)^2 - 2\mu_2\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_1)^2 - 2\mu_1\Big]\Big]\Big\}\Phi(x)\frac{\partial}{\partial u}\end{aligned} \tag{46}$$

which yields the new invariant solution $\overline{\omega}_1(t, x, y)$.

For solution (41), $\omega(t, x, y) = e^t\Phi(x)$,where $\Phi(x)$ is determined by differential equation (42). In this case, the new solution $\overline{\omega_2}$ is given by the relation:

$$\begin{aligned}[X_\omega, X_5] \equiv \overline{\omega_2}\frac{\partial}{\partial u} = \frac{e^t}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_1)^2\ln y - \rho\sigma_1\sigma_2\ln x + \\ \frac{t}{2}\Big[(\sigma_1)^2\Big[(\sigma_2)^2 - 2\mu_2\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_1)^2 - 2\mu_1\Big]\Big]\Big\}\Phi(x)\frac{\partial}{\partial u}\end{aligned} \tag{47}$$

For solution (43), $\omega(t, x, y) = y\Phi(x)$,where $\Phi(x)$ is determined by differential equation (44). In this last case, the property (45) is particularized as follows:

$$\begin{aligned}[X_\omega, X_5] \equiv \overline{\omega_3}\frac{\partial}{\partial u} = \Big\{-t + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_1)^2\ln y - \rho\sigma_1\sigma_2\ln x + \\ \frac{t}{2}\Big[(\sigma_1)^2\Big[(\sigma_2)^2 - 2\mu_2\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_1)^2 - 2\mu_1\Big]\Big]\Big\}y\Phi(x)\frac{\partial}{\partial u}\Big\}\end{aligned} \tag{48}$$

Remark 2: This procedure can be easily repeated by choosing the previous new solutions $\{\overline{\omega}_i, i = 1, 2, 3\}$ as starting solutions in (45). Furthermore, new solutions can be provided, by replacing X_5 with any linear combination of the symmetry operators (15). The iteration of this procedure leads to an infinite set of distinctly different solutions for the equation (2).

4. Inverse symmetry problem for generalized Black-Scholes models

The number of equations and unknown functions which appear in the determining system (5) is relatively large. Two approaches are possible: (i) to find the symmetries of a given evolutionary equation, which means to choose concrete forms for $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$, $F(x,y)$, $G(x,y,u)$ and to solve the differential system in order to find the coefficient functions of the general Lie operator $\xi(x,y,t,u)$, $\eta(x,y,t,u)$, $\phi(x,y,t,u)$; (ii) to solve the master system taking as unknown variables the functions $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $E(x,y)$, $F(x,y)$, $G(x,y,u)$ and imposing a concrete form for the symmetry group. The first approach represents the direct symmetry problem and it is the usual investigation followed in the study of Lie symmetries for a given dynamical system. The second approach defines the inverse symmetry problem and it is more special, allowing us to determine all equations which belong to the same class from the point of view of the symmetry group they admit.

In the previous sections we tackled with the approach (i). In this section we deal with the so-called inverse symmetry problem, the way (ii) mentioned before. We shall see which is the largest class of equations of form (3) admiting an imposed symmetry group. We shall choose the infinite-dimensional Lie algebra which corresponds to the following coefficient functions of the Lie operator (4):

$$\varphi = c_0 = const.,\ \xi = \eta = 0,\ \phi = ku + \omega(t,x,y); k = const. \tag{49}$$

Let us note that (49) generates a subalgebra of the algebra (15) with the following basic operators:

$$Y_1 = \frac{\partial}{\partial t},\ Y_2 = u\frac{\partial}{\partial u},\ Y_\omega = \omega(t,x,y)\frac{\partial}{\partial u} \tag{50}$$

where $\omega(t,x,y)$ represents an arbitrary solution of equation (3).

Our finding is that, solving under these conditions the determining system (5), the largest class of equations which admit (50) as Lie algebra is given by:

$$\begin{aligned} u_t \;=\;& A_1(x)A_2(y)u_{xy} + C(x)u_{2x} + D_1(x)\left[A_2(y)\right]^2 u_{2y} + E(x,y,t,u)u_y + \\ & F(x,y,t,u)u_x + \frac{k-1}{k}W(x,y)u \end{aligned} \tag{51}$$

for arbitrary functions $A_1(x)$, $A_2(y)$, $C(x)$, $D_1(x)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $W(x,y)$.

We have to mention that a special case of equation (51) is the $2D$ financial model suggested by Jacobs and Jones [12]:

$$\begin{aligned} u_t \;=\;& \frac{1}{2}a^2x^2u_{2x} + abcxyu_{xy} + \frac{1}{2}b^2y^2u_{2y} + \left[dx\ln\frac{y}{x} - ex^{(3/2)}\right]u_x + \\ & \left[fy\ln\frac{g}{y} - hy^{(1/2)}\right]u_y - xu \end{aligned} \tag{52}$$

It admits the infinite-dimensional Lie algebra (50) for any arbitrary parameters a, b, c, d, e, f, g, h. Therefore, from the point of view of the group classification, the Jacobs-Jones equation can be generalized in the form (51).

Remark 3: We can come back now to the direct symmetry method applied in Section 2 and we can recover for the Jacobs-Jones equation (52) the whole associated Lie algebra. In this respect we have to introduce in our general determining system (5) some restrictions upon the model's parameters. For example, by imposing in (52) the additional conditions:

$$ab \neq 0,\ c \neq \{-1, 0, 1\},\ d = 0,\ f = 0,\ ah - bce = 0 \tag{53}$$

the Lie algebra (50) is extended with the following operators:

$$Y_3 = y\frac{\partial}{\partial y},\ Y_4 = 2ab^2(1-c^2)ty\frac{\partial}{\partial y} + [2bc\ln x - 2a\ln y + ab(b-ac)t]\, u\frac{\partial}{\partial u} \tag{54}$$

5. Concluding remarks

We investigated the generalized second order nonlinear equation (3), which includes many equations coming from finance, as for examples Black-Scholes and Jacobs-Jones equations. We applied the direct symmetry approach for computing Lie algebra and some classes of invariant solutions for the $2D$ BS model. Then we used the inverse symmetry method for deriving the most general class of equations observing the same symmetry group. The main results of our investigations are the following: (i) the Lie algebra of the $2D$ BS model is an infinite dimensional one, looking quite alike the algebra of the $1D$ model. It can be decomposed as in the relation (13), with the generators given by (11) and (12); (ii) we can use the Lie algebra in order to generate an infinite number of invariant solutions. The procedure is illustrated by deriving some classes of invariant solutions for one-dimensional, respectively for two-dimensional subalgebras of the whole Lie algebra (13) of the $2D$ BS model: (iii) the inverse symmetry approach has been helpful in finding the most general class of evolutionary second order PDEs which admit the infinite-dimensional Lie algebra spanned by symmetry operators (50). It has the form (51); (iv) a particular case of equation included in the class (51) is (52). It can be recognized as the $2D$ financial model suggested by Jacobs and Jones. For this model, we have shown that the dimension of the symmetry group depends essentially on the parameters of the model. More exactly, we applied again the direct symmetry approach and we obtained, for concrete conditions (53), an extension of the Lie algebra (50). Under other conditions as in (53) compatible to the determining symmetry system, it is possible to attend to the whole Lie group classification for the Jacobs-Jones model.

References

[1] S. Galam, *Physica* **A 336** (2004) 49.

[2] F. Black and M. Scholes, *J. Polit. Econ.* **81** (1973) 637.

[3] R.C. Merton, *J. Econ. Manag. Sci.* **4** (1973) 141.

[4] V. Yakovenko and J.B. Rosser, *Rev. Mod. Phys.* **81** (2009) 1703.

[5] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

[6] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.

[7] N.H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordrechet, 1985.

[8] W. Sinkala, P.G.L. Leach and J.G. O'Harac, *J. Differ. Equations* **244** (2008) 2820.

[9] C.F. Lo and C.H. Hui, *Quant. Financ.* **1** (2001) 73.

[10] Y.K. Kwok, *Mathematical Models of Financial Derivatives*, Springer-Verlag, 1998.

[11] R.Cimpoiasu and R.Constantinescu, *Nonlinear Anal-Theor* **73** (2010) 147.

[12] R.L. Jacobs and R.A. Jones, *A two factor latent variable model of the term structure of interest rates*, Simon Fraser University, Burnaby, Canada, 1986.

[13] J.P. Singh and S. Prabakaran, *EJTP* **5** (2008) 51.

[14] R.M. Edelstein and K.S. Govinder, *Nonlinear Anal-Real* **10** (2009) 3372.

[15] N.H. Ibragimov, *Handbook of Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, FL, 1996.