Modified gravity: walk through accelerating cosmology *

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Abstract

We review the accelerating (mainly, dark energy) cosmologies in modified gravity. Special attention is paid to cosmologies leading to finite-time future singularities

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in $F(R)$, $F(G)$ and $\mathcal{F}(R,G)$ modified gravities. The removal of the finite-time future singularities via addition of R^2-term which simultaneously unifies the early-time inflation with late-time acceleration is also briefly mentioned. Accelerating cosmology including the scenario unifying inflation with dark energy is considered in $F(R)$ gravity with Lagrange multipliers. In addition, we examine domain wall solutions in $F(R)$ gravity. Furthermore, covariant higher derivative gravity with scalar projectors is explored.

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1. Introduction

It is observationally implied that the current expansion of the universe is accelerating. Provided that the universe is homogeneous, two representative approaches to account for the current cosmic acceleration exist. The first is to assume the existence of the so-called dark energy whose pressure is negative (for a recent review, see, e.g., [1]). The second is to consider that a gravitational theory would be modified at the large distance scale. The simplest theory is $F(R)$ gravity (for reviews, see, for example, [2]).

In this paper, we examine the accelerating (dark energy) solutions of modified gravity which may produce future singularities. We concentrate on reviewing the results in Refs. [3, 4, 5, 6] on theoretical aspects of modified gravity theories with presenting dark energy components. In particular, we study the finite-time future singularities in $F(R)$, $F(G)$ and $\mathcal{F}(R,G)$ gravity theories [3, 7, 8], where R is the Ricci scalar, $G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ with $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ the Ricci tensor and the Riemann tensor, respectively, is the Gauss-Bonnet invariant, and $\mathcal{F}(R,G)$ is an arbitrary function of R and G. This is a generalized gravity theory including both $F(R)$ and $F(G)$ gravity theories. We also discuss the removal of the finite-time future singularities in $F(R)$ gravity via addition of R^2-term which simultaneously leads to the unification of early-time inflation with late-time acceleration [9]. In the frameworks of $F(G)$ or $\mathcal{F}(R,G)$ theory, the corresponding term may be different, of course [3]. We note that as related studies, the finite-time future singularities [10, 11, 12] and the realization of the phantom phase including the crossing of the phantom divide [13] have also been examined. Furthermore, the features of the finite-time future singularities in non-local gravity [14], modified teleparallel gravity [15] and its extended analysis in loop quantum cosmology (LQC) [16] has recently been investigated. In addition, dark energy in the context of $F(R)$ gravity with Lagrange multipliers [4] is considered. We also present domain wall solutions in $F(R)$ gravity [5]. Moreover, covariant higher derivative gravity with scalar projectors [6] is explained. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G_{\mathrm{N}}$ by $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2 = 1$ with the Planck mass of $M_{\mathrm{Pl}} = G_{\mathrm{N}}^{-1/2} = 1.2 \times 10^{19}\mathrm{GeV}$.

The paper is organized as follows. In Section 2, we explore accelerating cosmologies leading to the finite-time future singularities in $F(R)$, $F(G)$

and $\mathcal{F}(R,G)$ gravity theories. In Section 3, we study dark energy in the framework of $F(R)$ gravity with Lagrange multipliers. In Section 4, we examine domain wall solutions in $F(R)$ gravity. In Section 5, we investigate covariant higher derivative gravity with scalar projectors. Finally, conclusions are presented in Section 6.

2. Finite-time future singularities in $F(R)$, $F(G)$ and $\mathcal{F}(R,G)$ gravity theories

2.1. $\mathcal{F}(R,G)$ gravity

The action of $\mathcal{F}(R,G)$ gravity is $S = \int d^4x\sqrt{-g}\left[\mathcal{F}(R,G)/\left(2\kappa^2\right) + \mathcal{L}_{\mathrm{M}}\right]$, where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{\mathrm{M}}$ is the matter Lagrangian. This is a generic theory including both $F(R)$ and $F(G)$ gravities. We take the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t)\sum_{i=1,2,3}\left(dx^i\right)^2$. The Hubble parameter is given by $H = \dot{a}/a$, where the dot denotes the time derivative of $\partial/\partial t$. In the FLRW background, with the gravitational field equations we find that the effective (i.e., total) energy density and pressure of the universe read $\rho_{\mathrm{eff}} = 3\kappa^{-2}H^2$ and $P_{\mathrm{eff}} = -\kappa^{-2}\left(2\dot{H} + 3H^2\right)$, respectively. For the action in Eq. (1), we obtain

$$\rho_{\mathrm{eff}} \equiv \frac{1}{\mathcal{F}_{,R}}\left\{\rho_{\mathrm{M}} + \frac{1}{2\kappa^2}\left[\left(\mathcal{F}_{,R}R - \mathcal{F}\right) - 6H\dot{\mathcal{F}}_{,R} + G\mathcal{F}_{,G} - 24H^3\dot{\mathcal{F}}_{,G}\right]\right\}\,, \quad (1)$$

$$\begin{aligned} P_{\mathrm{eff}} \equiv \frac{1}{\mathcal{F}_{,R}}\Bigg\{P_{\mathrm{M}} + \frac{1}{2\kappa^2}\Big[&-\left(\mathcal{F}_{,R}R - \mathcal{F}\right) + 4H\dot{\mathcal{F}}_{,R} + 2\ddot{\mathcal{F}}_{,R} - G\mathcal{F}_{,G} \\ &+ 16H\left(\dot{H} + H^2\right)\dot{\mathcal{F}}_{,G} + 8H^2\ddot{\mathcal{F}}_{,G}\Big]\Bigg\}\,, \end{aligned} \quad (2)$$

where $\mathcal{F}_{,R} \equiv \partial\mathcal{F}(R,G)\partial R$ and $\mathcal{F}_{,G} \equiv \partial\mathcal{F}(R,G)\partial G$, and ρ_{M} and P_{M} are the energy density and pressure of matter (which has been assumed to be a perfect fluid).

2.2. Finite-time future singularities

Provided that the Hubble parameter is written as

$$H = \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{\beta}} + H_{\mathrm{s}}\,, \quad (3)$$

where $h_{\mathrm{s}}(> 0)$, $t_{\mathrm{s}}(> 0)$, $H_{\mathrm{s}}(\geq 0)$, and $\beta(\neq 0)$ are constants, t_{s} is the time when a finite-time future singularity occurs, and $0 < t < t_{\mathrm{s}}$. In what follows, we consider the case of $H_{\mathrm{s}} = 0$. We note that even if $\beta < 0$ and β is a non-integer value, in the limit $t \to t_{\mathrm{s}}$ some derivative of H diverges and hence the scalar curvature becomes infinity [10]. Moreover, since the case of $\beta = 0$ leads to a de Sitter space, we suppose $\beta \neq 0$.

The finite-time future singularities are classified into four types [17]. Type I ("Big Rip" [18]): In the limit $t \to t_{\rm s}$, $a \to \infty$, $\rho_{\rm eff} \to \infty$ and $|P_{\rm eff}| \to \infty$. The case that $\rho_{\rm eff}$ and $P_{\rm eff}$ are finite values at $t = t_{\rm s}$ [19] is included. This happens for $\beta = 1$ and $\beta > 1$. In this paper, we regard the singularities for $\beta = 1$ as "Big Rip" and those for $\beta > 1$ as "Type I". (ii) Type II ("sudden" [20]): In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to \rho_{\rm s}$ and $|P_{\rm eff}| \to \infty$. This occurs for $-1 < \beta < 0$. (iii) Type III: In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to \infty$ and $|P_{\rm eff}| \to \infty$. This appears for $0 < \beta < 1$. (iv) Type IV: In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to 0$, $|P_{\rm eff}| \to 0$, and higher derivatives of H diverge. The case that $\rho_{\rm eff}$ and/or $|P_{\rm eff}|$ become finite values at $t = t_{\rm s}$ is also included. This is realized if $\beta < -1$ but β is not any integer number. Here, $a_{\rm s}(\neq 0)$ and $\rho_{\rm s}$ are constants.

2.3. $F(R)$ gravity with finite-time future singularities

By taking $\mathcal{F}(R, G) = F(R)$, the action in Eq. (1) becomes that of $F(R)$ gravity. With the method to reconstruct modified gravity [10, 21, 22], for the Hubble parameter to be represented in Eq. (3) we explore $F(R)$ gravity models in which finite-time future singularities can appear. By introducing two proper functions $P(\phi)$ and $Q(\phi)$ of a scalar field ϕ, which we regard as the cosmic time t, we rewrite the term $\mathcal{F}(R, G) = P(t)R + Q(t)$ in the action in Eq. (1). In this case, by varying the action with respect to t we acquire $(dP(t)/dt)\, R + dQ(t)/dt = 0$. In principle, by solving this equation we have the relation $t = t(R)$. If we substitute it into the above form of $\mathcal{F}(R, G) = P(t)R + Q(t)$, we find $F(R) = P(t = t(R))R + Q(t = t(R))$ and hence the original action is found again. We express the scale factor as $a(t) = \bar{a} \exp\left(\bar{g}(t)\right)$ with $\bar{a}$ a constant and $\bar{g}(t)$ a proper function. Here, we neglect the contribution from matter because when the finite-time future singularities appears, the energy density of dark energy components are completely dominant over that of matter. In this case, the gravitational field equations yield

$$\ddot{P}(t) - \dot{\bar{g}}(t)\dot{P}(t) + 2\ddot{\bar{g}}(t)P(t) = 0\,, \quad Q(t) = -6\left[\left(\dot{\bar{g}}(t)\right)^2 P(t) + \dot{\bar{g}}(t)\dot{P}(t)\right]. \tag{4}$$

Accordingly, if we find the solutions $P(t)$ and $Q(t)$ of these equations, by plugging those into $F(R) = P(t)R + Q(t)$ with $t = t(R)$ we obtain the concrete form of $F(R)$. We acquire the followings consequences.

(a) $\beta = 1$ [Big Rip]: For $h_{\rm s} > 5 + 2\sqrt{6}$ or $h_{\rm s} < 5 - 2\sqrt{6}$, $F(R) \propto R^q$ with $q \equiv (1/4)\left(3 + h_{\rm s} + \sqrt{h_{\rm s}^2 - 10h_{\rm s} + 1}\right)$, whereas if $5 - 2\sqrt{6} < h_{\rm s} < 5 + 2\sqrt{6}$, $F(R) \propto R^{(h_{\rm s}+1)/4} \times$ (Oscillating part).

(b) $\beta > 1$ [Type I]: $F(R) \propto \exp\left\{\left(h_{\rm s}/\left[2\left(\beta - 1\right)\right]\right)\left(\frac{R}{12h_{\rm s}}\right)^{(\beta-1)/(2\beta)}\right\} R^{-1/4} \times$ (Oscillating part).

(c) $0 < \beta < 1$ [Type III]: $F(R) \sim \exp\left[\frac{h_{\rm s}}{2(\beta-1)}\left(-6\beta h_{\rm s}R\right)^{(\beta-1)/(\beta+1)}\right] R^{7/8}$.

(d) $\beta < 0$ [Type II ($-1 < \beta < 0$) and Type IV ($\beta < -1$ but β is not an integer)]: $F(R) \sim \left(-6h_{\rm s}\beta R\right)^{\left(\beta^2+2\beta+9\right)/[8(\beta+1)]} \exp\left[\frac{h_{\rm s}}{2(\beta-1)}\left(-6h_{\rm s}\beta R\right)^{(\beta-1)/(\beta+1)}\right]$.

Here, "$\sim$" means the asymptotic behavior in the limit $t \to t_{\mathrm{s}}$.

It is remarkable that adding R^2-term to such a theory, one removes future singularity. (Note that R^2 gravity was proposed as inflationary model in Ref. [23] (celebrated Starobinsky inflation) and was used for the first unified inflation-dark energy modified gravity proposed in Ref. [9]). Hence, we not only remove singularities by adding R^2 term but also unify the dark energy era with inflation in such a way (for realistic models of such unification, see [2]). Several viable $F(R)$ gravity models which unify inflation with dark energy and do not contain the finite-time future singularities are listed below [2]:

$$F(R) = R + \frac{b_1 R^{2l} - b_2 R^l}{1 + b_3 R^l} + b_4 R^2 \,, \tag{5}$$

$$F(R) = R - 2\Lambda \left[1 - \mathrm{e}^{-R/(b_5\Lambda)}\right] - \Lambda_{\mathrm{i}} \left[1 - \mathrm{e}^{-(R/R_{\mathrm{i}})^{\varsigma}}\right] + b_6 \tilde{R}_{\mathrm{i}}^{-(\tau-1)} R^{\tau} \,, \tag{6}$$

with b_j ($j = 1, \ldots, 4$), $b_5(> 0)$, $b_6(> 0)$ and l constants. In Eq. (6), $\tau(> 1)$ is a natural number, ς and $\tilde{R}_{\mathrm{i}}$ are constants, R_{i} and Λ_{i} are transition curvature and expected cosmological constant at the inflationary stage, respectively [24].

2.4. $F(G)$ gravity with finite-time future singularities

With the same method as in $F(R)$ gravity in Section 2.3, it is possible to execute the reconstruction of $F(G)$ gravity models in which the finite-time future singularities occur. The action of $F(G)$ gravity [25] is described by Eq. (1) with $\mathcal{F}(R, G) = R + F(G)$. In this case, the gravitational field equations yield

$$2\frac{d}{dt}\left(\dot{g}^2(t)\dot{P}(t)\right) - 2\dot{g}^3(t)\dot{P}(t) + \ddot{g}(t) = 0 \,, \quad Q(t) = -24\dot{g}^3(t)\dot{P}(t) - 6\dot{g}^2(t) \,. \tag{7}$$

The results are as follows [3].

(a) $\beta = 1$ [Big Rip]: For $h_{\mathrm{s}} \neq 1$, $F(G) = \left\{\sqrt{6h_{\mathrm{s}}^3(1+h_{\mathrm{s}})} / \left[h_{\mathrm{s}}(1-h_{\mathrm{s}})\right]\right\} \sqrt{G} + c_1 G^{(h_{\mathrm{s}}+1)/4} + c_2 G$ with c_1 and c_1 constants. If $h_{\mathrm{s}} = 1$, $F(G) = \frac{\sqrt{3}}{2}\sqrt{G}\ln(\gamma G)$ with $\gamma(> 0)$ a positive constant.

(b) $\beta > 1$ [Type I]: $F(G) = -\sqrt{6}\sqrt{G}$.

(c) $0 < \beta < 1$ [Type III], $-1/3 < \beta < 0$ [Type II], $-1 < \beta < -1/3$ [Type II] and $\beta < -1$ (but β is not integer) [Type IV]: $F(G) = 6h_{\mathrm{s}}^2(3\beta+1)(\beta+1)^{-1}\left[|G| / \left(24h_{\mathrm{s}}^3|\beta|\right)\right]^{2\beta/(3\beta+1)}$.

(d) $\beta = -1/3$ [Type II] (this is a special value in this case): $F(G) \simeq \left[1/\left(4\sqrt{6}h_{\mathrm{s}}^3\right)\right] G \left(G + 8h_{\mathrm{s}}^3\right)^{1/2} + \left(2/\sqrt{6}\right)\left(G + 8h_{\mathrm{s}}^3\right)^{1/2}$. We remark that the finite-time future singularities appearing in the limit $G \to \pm\infty$ can be removed by the additional term $d_1 G^{\varrho}$, where $d_1(\neq 0)$ is a constant, and $\varrho > 1/2$ and $\varrho \neq 1$. Furthermore, the finite-time future singularities emerging in the limit $G \to 0^-$ can be cured by adding the term $d_1 G^{\varrho}$, where $\varrho(\leq 0)$ is an integer [3].

2.5. $\mathcal{F}(R,G)$ gravity with finite-time future singularities

Using the similar procedure in Section 2.3, we reconstruct the form of $\mathcal{F}(R,G)$ leading to the finite-time future singularities. With proper functions $P(\phi)$, $Z(\phi)$ and $Q(\phi)$ of a scalar field ϕ, which we identify with t, we represent the term $\mathcal{F}(R,G)$ in the action in Eq. (1) as $P(t)R+Z(t)G+Q(t)$. Varying this action with respect to t, we find $(dP(t)/dt)\,R+(dZ(t)/dt)\,G+dQ(t)/dt=0$. By solving this equation, we obtain $t=t(R,G)$. Combining this and the above representation $P(t)R+Z(t)G+Q(t)$, we acquire $\mathcal{F}(R,G)=P(t)R+Z(t)G+Q(t)$. It follows from the gravitational field equations, the conservation law, $a(t)=\bar{a}\exp(\bar{g}(t))$, and $H(t)=\dot{\bar{g}}(t)$ that

$$\frac{d^2P(t)}{dt^2}+4\dot{\bar{g}}^2(t)\frac{d^2Z(t)}{dt^2}-\dot{\bar{g}}(t)\frac{dP(t)}{dt}$$
$$+4\left(2\dot{\bar{g}}\ddot{\bar{g}}-\dot{\bar{g}}^3(t)\right)\frac{dZ(t)}{dt}+2\ddot{\bar{g}}(t)P(t)=0\,, \tag{8}$$
$$Q(t)=-6\left(4\dot{\bar{g}}^3(t)\frac{dZ(t)}{dt}-\dot{\bar{g}}^2(t)P(t)-\dot{\bar{g}}(t)\frac{dP(t)}{dt}\right)\,. \tag{9}$$

For $P(t)\neq 0$, $\mathcal{F}(R,G)$ can be described as $\mathcal{F}(R,G)=R\tilde{g}(R,G)+\tilde{f}(R,G)$ with $\tilde{g}(R,G)(\neq 0)$ and $\tilde{f}(R,G)$ generic functions of R and G. We show the results.

(a) $\beta=1$ [Big Rip]: For $0<h_{\mathrm{s}}<5-2\sqrt{6}$ or $h_{\mathrm{s}}>2+\sqrt{6}$, we find $\mathcal{F}(R,G)=\alpha_1 R^{q_+}+\alpha_2 R^{q_-}+\delta G^{(h_{\mathrm{s}}+1)/4}$ with $q_\pm\equiv(1/4)\left(3+h_{\mathrm{s}}\pm\sqrt{h_{\mathrm{s}}^2-10h_{\mathrm{s}}+1}\right)$. Here, α_1, α_2 and δ are constants. There also exists the following model: $\mathcal{F}(R,G)=\frac{\alpha}{(\tilde{f}(R,G))^{x+2}}R+\frac{\delta}{(\tilde{f}(R,G))^{x}}G-\frac{6h_{\mathrm{s}}}{(\tilde{f}(R,G))^{x+4}}\left[4h_{\mathrm{s}}^2\delta x+\alpha(x+2+h_{\mathrm{s}})\right]$, where $\tilde{f}(R,G)=\left\{\frac{-\alpha(x+2)R\pm\sqrt{\alpha^2(x+2)^2R^2+24h_{\mathrm{s}}[4h_{\mathrm{s}}^2\delta x+\alpha(x+2+h_{\mathrm{s}})](x+4)\delta xG}}{2\delta xG}\right\}^{1/2}$. Here, α and x are constants.

(b) $\beta>1$ [Type I]: $\mathcal{F}(R,G)=-4h_{\mathrm{s}}^2\lambda f(R,G)R+\lambda\left(f(R,G)\right)^{1+2\beta}G+24h_{\mathrm{s}}^4\lambda\left(f(R,G)\right)^{1-2\beta}$ with $\tilde{f}(R,G)=\left[\frac{h_{\mathrm{s}}^2R+\sqrt{h_{\mathrm{s}}^4R^2+6h_{\mathrm{s}}^4(4\beta^2-1)G}}{(1/2+\beta)G}\right]^{1/(2\beta)}$, where λ is a constant.

(c) $\beta<1$ [Type II ($-1<\beta<0$), Type III ($0<\beta<1$), and Type IV ($\beta<-1$ but β is not an integer)]: $\mathcal{F}(R,G)=R+(3/2)\,(G/R)$. In Ref. [3], it has been examine that the finite-time future singularities can be removed by the term $R^{\vartheta_1}G^{\vartheta_2}$, where $\vartheta_1(>0)$ and $\vartheta_2(>0)$ are positive integers.

3. Dark energy from $F(R)$ gravity with the Lagrange multipliers

In this section, we study $F(R)$ gravity with the Lagrange multiplier field. With $F_1(R)$ and $F_2(R)$ arbitrary functions of R, the action is expressed as

$$S = \int d^4x\sqrt{-g}\left[F_1(R) - \lambda_{\rm L}\left(\frac{1}{2}\partial_\mu R\partial^\mu R + F_2(R)\right)\right], \tag{10}$$

where $\lambda_{\rm L}$ is the Lagrange multiplier field and yields a constraint equation $(1/2)\,\partial_\mu R\partial^\mu R + F_2(R) = 0$. The variation of the action in Eq. (10) with respect to $g_{\mu\nu}$ leads to the gravitational field equation as

$$\frac{1}{2}g_{\mu\nu}F_1(R) + \frac{1}{2}\lambda_{\rm L}\partial_\mu R\partial_\nu R + \left(-R_{\mu\nu} + \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right) \times\left[\frac{dF_1(R)}{dR} - \lambda_{\rm L}\frac{dF_2(R)}{dR} - \nabla^\mu\left(\lambda_{\rm L}\nabla_\mu R\right)\right] = 0\,, \tag{11}$$

where ∇_μ is the covariant derivative and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian. For the de Sitter space-time, which realize the current cosmic accelerated expansion, i.e., the dark energy dominated stage, the scalar curvature is a positive constant value R_0 and hence the Ricci tensor becomes $R_{\mu\nu} = (1/4)\,R_0 g_{\mu\nu}$. In this case, from the above constraint equation and Eq. (11), we have $\lambda_{\rm L} = \left[-2F_1(R_0) + R_0\left(dF_1(R_0)/dR\right)\right]/\left[R_0\left(dF_2(R_0)/dR\right)\right]$. Moreover, in the flat FLRW background, the above constraint equation reads $-(1/2)\,\dot{R}^2 + F_2(R) = 0$. For $F_2(R) > 0$, this equation can be solved in terms of t as $t = \int^R dR/\sqrt{2F_2(R)}$. Provided that the form of $H(t)$ is given by the analysis of the observational data, $F_2(R)$ is able to be reconstructed so that the evolution of $H(t)$ can be reproduced. It follows from $R = 6\left[dH/dt + 2H^2\right]$ that $H(t)$ presents the evolution of $R = R(t)$, and by solving this equation inversely, we can find $t = t(R)$. Thus, we acquire $F_2(R) = (1/2)\left(dR/dt\right)^2$ with $t = t(R)$. We note that $F_1(R)$ is an arbitrary function of R. As an example, we consider $H(t) = h_0/t$ with $h_0 > 1$ leading to $a(t) = a_0 t^{h_0}$, where h_0 and a_0 are constants. In this case, the accelerated expansion of the universe or power-law inflation happens. We have $R = 6h_0\left(-1 + 2h_0\right)/t^2$, from which we also acquire $t = \sqrt{6h_0\left(-1 + 2h_0\right)/R}$. Using these relations, we obtain $F_2(R) = R^3/\left[12h_0\left(-1 + 2h_0\right)\right]$. As another example, we examine the case that R is described by $R = (R_-/2)\left(1 - \tanh\omega t\right) + (R_+/2)\left(1 + \tanh\omega t\right)$ with $R_\pm(> 0)$ and $\omega(> 0)$ positive constants. In the limit $t \to \pm\infty$, $R \to \pm R_\pm$, and therefore the universe asymptotically approaches the de Sitter space-time. In this case, we can regard that in the limit $t \to -\infty$, inflation in the early universe occurs, whereas that in the limit $t \to +\infty$, the late-time cosmic acceleration happens. We also have $F_2(R) = (1/8)\left(R_- - R_+\right)^2\omega^2\left[1 - \left(R_- + R_+ - 2R\right)^2/\left(R_- - R_+\right)^2\right]^2$. As a consequence, for the above R, this $F(R)$ gravity model with the constraint originating from the Lagrange multiplier can be a unified scenario between inflation and dark energy era, although it should carefully be studied whether the reheating stage after inflation can be realized. Furthermore, $F_1(R)$ does not affect cosmological evolution of the universe and influences only the correction of the Newton law. Thus, cosmology is determined only by the form of $F_2(R)$.

To explore the Newton law, we take $F_1(R) = R/\left(2\kappa^2\right)$ as the Einstein-Hilbert term and add matter. In this case, for $\lambda_{\mathrm{L}} = 0$, from Eq. (11) we find the Einstein equation $\left[R_{\mu\nu} - (1/2)\, g_{\mu\nu} R\right] = \kappa^2 T^{(\mathrm{M})}_{\mu\nu}$ with $T^{(\mathrm{M})}_{\mu\nu}$ the energy-momentum tensor of matter Its trace equation reads $R = -\kappa^2 T^{(\mathrm{M})}$, where $T^{(\mathrm{M})}$ is the trace of $T^{(\mathrm{M})}_{\mu\nu}$. Moreover, the constraint equation is given by $\left(\kappa^4/2\right) \partial_\mu T \partial^\mu T + F_2\left(-\kappa^2 T\right) = 0$. Since this is not always met, we should modify the constraint equation as $(1/2)\, \partial_\mu R \partial^\mu R + F_2(R) - \left(\kappa^4/2\right) \partial_\mu T \partial^\mu T^{(\mathrm{M})} - F_2\left(-\kappa^2 T\right) = 0$. Thus, this implies that the action with the constraint coming from the Lagrange multiplier field and matter should be described by

$$\begin{aligned} S \;=\; & \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - \lambda_{\mathrm{L}} \left[\frac{1}{2} \partial_\mu R \partial^\mu R + F_2(R) - \frac{\kappa^4}{2} \partial_\mu T^{(\mathrm{M})} \partial^\mu T^{(\mathrm{M})} \right.\right. \\ & \left.\left. - F_2\left(-\kappa^2 T^{(\mathrm{M})}\right)\right] + \mathcal{L}_{\mathrm{M}} \right\}, \end{aligned} \tag{12}$$

For the case of the vacuum such that $T^{(\mathrm{M})} = 0$, the constraint equation is $(1/2)\, \partial_\mu R \partial^\mu R + F_2(R) - F_2\,(0) = 0$. If $F_2\,(0) = 0$, e.g., the first example of $F_2(R) = R^3/\left[12 h_0 \left(-1 + 2h_0\right)\right]$ shown above, this is equivalent to the constraint equation derived from the action in Eq. (10). In this case, there exist two types of the solutions in the constraint equation $-\,(1/2)\, \dot{R}^2 + F_2(R) = 0$. One is $R = 0$ and the other is presented by $t = \int^R dR/\sqrt{2F_2(R)}$. On the small scales of, e.g., the solar system and galaxies, the solution would be the first solution of $R = 0$ so that the Newton law can be recovered. On the other hand, in the bulk of the universe, the solution should be $t = \int^R dR/\sqrt{2F_2(R)}$ in order that the cosmic evolution can be realized. It is not so clear whether the first solution on the small scales of the solar system and galaxies and the second one in the bulk universe can be connected in the intermediate scales.

4. Domain wall solutions in $F(R)$ gravity

In this section, we investigate a static domain wall solution and reconstruct an $F(R)$ gravity model with realizing it [5].

4.1. Static domain wall solution in a scalar field theory

To begin with, we study a static domain wall solution in a scalar field theory. We suppose that the following $D = d + 1$ dimensional warped metric $ds^2 = dy^2 + \mathrm{e}^{u(y)} \sum_{\mu,\nu=0}^{d-1} \hat{g}_{\mu\nu} dx^\mu dx^\nu$, and that the scalar field only depends on y. In this background, the metric in the d-dimensional Einstein manifold is $\hat{g}_{\mu\nu}$, defined by $\hat{R}_{\mu\nu} = \left[(d-1)/l^2\right]\hat{g}_{\mu\nu}$. In addition, for $1/l^2 > 0$, the space is the de Sitter one, for $1/l^2 > 0$, it is the anti-de Sitter, and for $1/l^2 = 0$, it is the flat. Following the procedure proposed in Ref. [26], it has been demonstrated that a static domain wall solution can exist in a scalar field theory [5] (a developed study on a static domain wall solution in a scalar field theory has also been executed in Ref. [27]). We investigate the action $S = \int d^D x \sqrt{-g}\left[\left(R/2\kappa^2\right) - (1/2)\,\omega(\varphi)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{V}(\varphi)\right]$, where $\omega(\varphi)$ is a function of the kinetic term of a scalar field φ and $\mathcal{V}(\varphi)$ is a potential of φ. In the above D dimensional warped metric, with the (y,y) and (μ,ν) components of the Einstein equation, we obtain the expressions of $\omega(\varphi)$ and $\mathcal{V}(\varphi)$. Using these expressions, the energy density is described as $\rho_\varphi \equiv (1/2)\,\omega(\varphi)\left(\varphi'\right)^2 + \mathcal{V}(\varphi)$. As an example, we consider $u = u_0 \exp\left(-y^2/y_0^2\right)$, where u_0 and y_0 are constants. In this case, the distribution of ρ_φ reads $\rho_\varphi(y) = -\left[(d-1)/\left(2y_0^2\right)\right]\left[\left(2y^2/y_0^2\right) - 1\right]\exp\left(-y^2/y_0^2\right) + \left[(d-1)^2/l^2\right]\exp\left[-u_0\exp\left(-y^2/y_0^2\right)\right]$. Accordingly, the energy density of φ is localized at $y \sim 0$ and thus a domain wall is made. We note that a condition for ρ_φ to be localized is $u \to 0$ in the limit $|y| \to \infty$.

4.2. Reconstruction of the form of $F(R)$

In the D dimensional warped metric, the (y,y) component and the trace of (μ,ν) components of the gravitational field equation read

$$\frac{d-1}{2}u'\left(F_{,R}\right)' - \frac{d}{2}\left[u'' + \frac{1}{2}\left(u'\right)^2\right]F_{,R} - \frac{1}{2}F = \kappa^2 T_{yy}^{(\mathrm{M})}\,, \tag{13}$$

$$d\left(F_{,R}\right)'' + \frac{d\,(d-2)}{2}u'\left(F_{,R}\right)' + \left\{-\frac{d}{2}\left[u'' + \frac{d}{2}\left(u'\right)^2\right] + \frac{d\,(d-1)}{l^2}\mathrm{e}^{-u}\right\}F_{,R}$$
$$-\frac{d}{2}F = \kappa^2 \sum_{\mu,\nu=0}^{d-1} g^{\mu\nu}T_{\mu\nu}^{(\mathrm{M})}\,, \tag{14}$$

where the prime denotes the derivative with respect to y of d/dy, and $\left(F_{,R}\right)' \equiv dF_{,R}/dy$ and $\left(F_{,R}\right)'' \equiv d^2F_{,R}/dy^2$. We examine an explicit form of $F(R)$ with leading to a domain wall solution for the case that matter is absent. For the model $u = u_0\exp\left(-y^2/y_0^2\right)$, with the relation $R = -d\left\{u'' + \left[(1+d)/4\right]\left(u'\right)^2\right\} + \left[d\,(d-1)/l^2\right]\mathrm{e}^{-u}$, y can be described as a function of R, $y = y(R)$, and eventually we find $u = u(y(R))$. By plugging this equation into Eqs. (13) and (14) and eliminating y, Eqs. (13) and (14) can be expressed as differential equations in terms of $F(R)$. Here, it is enough to analyze Eq. (13) because Eq. (14) is not independent of Eq. (13).

As a result, Eq. (13) can be rewritten to

$$\Xi_1(R)\frac{d^2F(R)}{dR^2} + \Xi_2(R)\frac{dF(R)}{dR} - F(R) = 0\,, \tag{15}$$

$$\Xi_1(R) \equiv (d-1)\, u' \frac{dR}{dy} = (d-1)\left(\frac{dR}{dy}\right)^2 \frac{du(y(R))}{dR}\,, \tag{16}$$

$$\Xi_2(R) \equiv (-d)\left[u'' + \frac{1}{2}\left(u'\right)^2\right] = (-d)\left[\frac{d^2R}{dy^2}\frac{du(y(R))}{dR} + \left(\frac{dR}{dy}\right)^2 \frac{d^2u(y(R))}{dR^2} + \frac{1}{2}\left(\frac{dR}{dy}\right)^2\left(\frac{du(y(R))}{dR}\right)^2\right]. \tag{17}$$

To solve the above relation of R in terms of y, by defining $Y \equiv y^2/y_0^2$ and expanding exponential terms in the limit $Y = y^2/y_0^2 \ll 1$, we take only the first leading terms in terms of Y. We find $Y = y^2/y_0^2 \approx (R-\gamma_1)/\gamma_2$ with $\gamma_1 \equiv \left(2du_0/y_0^2\right) + d\,(d-1)/l^2$ and $\gamma_2 \equiv -d\left(u_0/y_0^2\right)[6 + (1+d)\,u_0] + \left[d\,(d-1)/l^2\right] u_0$, where γ_1 and γ_2 are constants. Finally, for $Y = y^2/y_0^2 \ll 1$, Eq. (15) can be described by $\left(d^2F(R)/dR^2\right) + \mathcal{C}\,(dF(R)/dR) + \mathcal{D}F(R) = 0$ with $\mathcal{C} \equiv \Xi_2^{(0)}/\Xi_1^{(0)}$ and $\mathcal{D} \equiv -1/\Xi_1^{(0)}$, where $\Xi_1^{(0)}$ and $\Xi_2^{(0)}$ constants described by the model parameters d, l, u_0 and y_0. We acquire a general solution of this equation as $F(R) = F_+ \mathrm{e}^{\lambda_+ R} + F_- \mathrm{e}^{\lambda_- R}$, where $\lambda_\pm \equiv (1/2)\left(-\mathcal{C} \pm \sqrt{\mathcal{C}^2 - 4\mathcal{D}}\right)$, and $F_\pm$ are arbitrary constants. Here, the subscriptions $\pm$ of $\lambda_\pm$ correspond to the sign "$\pm$" on the right-hand side of this equation. In the model $u = u_0 \exp\left(-y^2/y_0^2\right)$, at $y \sim 0$ the distribution of the energy density is localized and therefore a domain wall is realized as shown above. Consequently, for an exponential model of $F(R)$ gravity, a domain wall can appear at $y \sim 0$.

4.3. Effective (gravitational) domain wall

Next, with the reconstruction method [21, 22], we explore an effective (gravitational) domain wall in $F(R)$ gravity. With the same procedure as in Section 2.3, we study the action of $F(R)$ gravity given by $\mathcal{F}(R,G) = F(R)$. Using two proper functions $P(\psi)$ and $Q(\psi)$ of a scalar field ψ, we represent the term $\mathcal{F}(R,G) = P(\psi)R + Q(\psi)$. The variation over ψ yields $(dP(\psi)/d\psi)\,R + dQ(\psi)/d\psi = 0$. Solving this equation with respect to ψ leads to $\psi = \psi(R)$, by substituting which into the action in Eq. (1) with $\mathcal{F}(R,G) = P(\psi)R + Q(\psi)$, we acquire the action of $F(R)$ gravity as $F(R) = P(\psi(R))R + Q(\psi(R))$. In the D dimensional warped metric shown in Section 4, for the case that ψ depends only on y, it follows from the gravitational field equation with the choice of $\psi = y$ and $1/l^2 = 0$ (i.e., the flat space), we have

$$u'(\psi) = -\frac{2}{d-1}\left[\frac{P'(\psi)}{P(\psi)} + \frac{d}{d-1}\left(P(\psi)\right)^{1/(d-1)} \times \int d\psi \left(P(\psi)\right)^{-(2d-1)/(d-1)} \left(P'(\psi)\right)^2\right], \quad (18)$$

$$Q(\psi) = \frac{d(d-1)\left(u'(\psi)\right)^2}{4} P(\psi) + (d-1)u'(\psi)P'(\psi)\,, \quad (19)$$

where the prime denotes the derivative with respect to $\psi(= y)$ of $d/d\psi$. For a model $P(\psi) = (U(\psi))^{-2(d-1)}$ and $U(\psi) = U_0 \left(\psi^2 + \psi_0^2\right)^\chi$ with U_0, ψ_0 and χ constants, we acquire

$$u'(\psi) = \frac{2\chi\psi}{\psi^2+\psi_0^2} - \frac{32d\chi^2\psi^{4\chi-1}}{\left(\psi^2+\psi_0^2\right)^{2\chi}} \times \sum_{k=0}^{\infty} \frac{\Gamma(2\chi-1)}{(4\chi-1-2k)\,\Gamma(2\chi-1-k)\,k!}\left(\frac{\psi_0^2}{\psi^2}\right)^k . \quad (20)$$

In the range where $\psi = y$ is large, we take $\chi = -1/\left[4\left(4d-1\right)\right]$ and impose the boundary condition that in the limit $|y| = |\psi| \to \infty$, the universe asymptotically approaches flat as $u \to 0$. As a result, we obtain $u(\psi) = -\left\{1/\left[4\left(6d-1\right)\right]\right\}(\psi_0/\psi) + \mathcal{O}\left((\psi_0/\psi)^3\right)$. From this expression, we see that $u(\psi)$ performs a non-trivial behavior at $\psi = y \sim 0$. Hence, it can be considered that an effective (gravitational) domain wall could appear at $y = 0$. Moreover, by using the representation $u'(\psi) = (4U'(\psi)/U(\psi)) - \left(8d/U(\psi)^2\right)\int d\psi \left(U'(\psi)\right)^2$, we acquire an integration expression of $u(\psi)$ as

$$u(\psi) = 8\chi \int_{-\infty}^{\psi} d\psi \frac{\psi}{\psi^2+\psi_0^2} - 32d\chi^2 \int_{-\infty}^{\psi} d\psi \frac{1}{\left(\psi^2+\psi_0^2\right)^{2\chi}} \int_0^{\psi} d\tilde{\psi}\left(\tilde{\psi}^2+\psi_0^2\right)^{2(\chi-1)}\tilde{\psi}^2 . \quad (21)$$

In Ref. [5], it has numerically been verified that there exists a local maximum of $u(\psi)$ at $\psi = y \sim 0$, and thus an effective (gravitational) domain wall could be realized at $y = 0$. Also, there occurs such a qualitative behavior of $u(\psi)$ in terms of ψ regardless of the values of the model parameters. In addition, we mention that for $U(\psi) = U_0\sqrt{\psi^2+\psi_0^2}$, i.e., $\chi = 1/2$, there exists an analytic solution

$$u(\psi) = 2\left(1-2d\right)\ln\left(\psi^2+\psi_0^2\right) + 4d\left(\arctan\left(\frac{\psi}{\sqrt{\psi_0^2}}\right)\right)^2 + \mathcal{C}\,, \quad (22)$$

with $\mathcal{C}$ an integration constant. In this case, for the region of a small amplitude of ψ, it is considered that the distribution of the energy density is localized, so that an effective (gravitational) domain wall could be made.

As a demonstration, we reconstruct an explicit $F(R)$ form for $u(\psi)$ in Eq. (22), although only in the region of a small amplitude of ψ, the distribution of the energy density could be regarded as an effective (gravitational) domain wall. From $P'(\psi)R + Q'(\psi) = 0$ and Eq. (19), we have

$$R = -\frac{Q'(\psi)}{P'(\psi)} = -\frac{(d-1)}{2P'(\psi)} \left(du'(\psi)u''(\psi) + 2u''(\psi)P'(\psi) + 2u'(\psi)P''(\psi) \right) . \tag{23}$$

Solving this equation, an analytic relation $\psi = \psi(R)$ can be found. With this relation, we acquire $F(R) = P(\psi(R))R + Q(\psi(R))$. In this case, we have $P(\psi) = (U_0\psi_0)^{-2(d-1)} \left(1 + \bar{Y}\right)^{-2(d-1)}$. For $\bar{Y} \equiv \psi^2/\psi_0^2 \ll 1$, by expanding Eq. (19) in terms of $\bar{Y}$ and taking the leading terms, we find $R = \mathcal{R}_0 + \mathcal{R}_1\bar{Y}$. Furthermore, from Eq. (19) we also obtain $Q = \mathcal{Q}_1\bar{Y} + \mathcal{Q}_2\bar{Y}^2$. Here, $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are constants and these are written by using the model parameters d, U_0 and ψ_0. Moreover, with $R = \mathcal{R}_0 + \mathcal{R}_1\bar{Y}$ we describe $\bar{Y} = \bar{Y}_0 + \bar{Y}_1 R$, where $\bar{Y}_0 \equiv -\mathcal{R}_0/\mathcal{R}_1$ and $\bar{Y}_1 \equiv 1/\mathcal{R}_1$. $P(\psi)$ can also be expanded as $P(\psi) \approx (U_0\psi_0)^{-2(d-1)} \left\{1 - (d-1)\bar{Y} + \left[d\,(d-1)/2\right]\bar{Y}^2\right\}$. By plugging this relation and $Q = \mathcal{Q}_1\bar{Y} + \mathcal{Q}_2\bar{Y}^2$ with $\bar{Y} = \bar{Y}_0 + \bar{Y}_1 R$ into $F(R) = P(\psi(R))R + Q(\psi(R))$ and taking terms of order of R^2, we find $F(R) = \mathcal{F}_0 + \mathcal{F}_1 R + \mathcal{F}_2 R^2$, where $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are constants and these are represented by the model parameters d, U_0 and ψ_0. The above explicit form of $F(R)$ has been derived for $\bar{Y} = \psi^2/\psi_0^2 \ll 1$. Hence, it follows from $R = \mathcal{R}_0 + \mathcal{R}_1\bar{Y}$ that this $F(R)$ form can be considered to correspond to the one for $R \sim \mathcal{O}(1)$ if $\mathcal{R}_0 \sim \mathcal{O}(1)$. Therefore, when we choose $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$, we have $F(R) = R + \mathcal{F}_2 R^2$. In this case, for the small curvature limit, $F(R)$ approaches R, i.e., general relativity, asymptotically. As a result, if $u(\psi)$ is given by Eq. (22) in which an effective (gravitational) domain wall can be realized, an explicit form of $F(R)$ is expressed as a power-law model. We state the difference between the domain walls in Sections 4.2 and 4.3. A pure gravitational effect yields an effective (gravitational) domain wall in Section 4.3, but a scalar field makes a static domain wall solution explored in Section 4.1. In Section 4.2, the deviation of $F(R)$ gravity from general relativity is equivalent to matter geometrically, i.e., a scalar field in Section 4.1.

5. Covariant higher derivative gravity with scalar projectors

It is considered that a covariant gravity which is power-counting renormalizable would be higher derivative theory, e.g., models in Ref. [28]. Higher derivative gravity is very well known to be renormalizable multiplicatively (for a review, see, for example, [29]). However, in general, such a higher derivative theory cannot keep the unitarity. To retain it, the so-called

Hořava gravity [30] has been proposed. In this section, we make the formulation for covariant higher derivative gravity with Lagrange multiplier constraint as well as scalar projectors. In particular, we construct a gravity theory with the Lorentz symmetry and/or the full general covariance in the action, although these symmetry and/or covariance is spontaneously broken. In such a theory, the propagator of the graviton in the ultraviolet (UV) region can be improved better, whereas there appears no extra mode such as a scalar one.

5.1. Model

We explore the following action with the Lagrange multiplier field $\lambda_{\rm L}$ [4, 31] as well as the scalar field Φ: $S_{\rm L} = -\int d^4x\sqrt{-g}\lambda_{\rm L}\left[(1/2)\,\partial_\mu\Phi\partial^\mu\Phi + \bar{W}\right]$, where μ and ν run $0,1,2,3$ and the 0 component denotes the time t as $\partial_0 \equiv \partial/\partial t$. From this action, we find a constraint equation $(1/2)\,\partial_\mu\Phi\partial^\mu\Phi + \bar{W} = 0$. This means that the vector quantity $(\partial_\mu\Phi)$ is time-like one. Hence, this breaks the Lorentz symmetry and/or the full general covariance spontaneously. For simplicity, we assume that $\bar{W}$ is a constant, although this assumption is not necessary for the symmetry and/or covariance to spontaneously be broken. Furthermore, the direction of time can be taken so that it should be parallel to the vector quantity $(\partial_\mu\Phi)$. In this case, we have $(1/2)\,(d\Phi/dt)^2 = \bar{W}$, from which we have $\Phi = \sqrt{2\bar{W}}t$. Accordingly, the spatial region is a hypersurface with a constant Φ because the hypersurface becomes orthogonal to the vector quantity $(\partial_\mu\Phi)$. In the flat space-time, we examine the perturbations as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu}$ the Minkowski metric and $h_{\mu\nu}$ corresponds to the fluctuations, i.e., the deviation of $g_{\mu\nu}$ from the Minkowski background $\eta_{\mu\nu}$. A projection operator is defined as ${\mathcal{P}_\mu}^\nu \equiv {\delta_\mu}^\nu + (\partial_\mu\Phi\partial^\nu\Phi)\,/\,\left(2\bar{W}\right)$ with ${\mathcal{P}_0}^\mu = 0$. As a result, the action describing a higher derivative gravity with scalar projector which is covariant and power-counting renormalizable is expressed as

$$S_{2n+2} = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \zeta\left[\left(\partial^\mu\Phi\partial^\nu\Phi\nabla_\mu\nabla_\nu - \partial_\mu\Phi\partial^\mu\Phi\nabla^\rho\nabla_\rho\right)^n {\mathcal{P}_\alpha}^\mu {\mathcal{P}_\beta}^\nu\right.\right.$$
$$\left.\times\left(R_{\mu\nu} - \frac{1}{2\bar{W}}\partial_\rho\Phi\nabla^\rho\nabla_\mu\nabla_\nu\Phi\right)\right]\left[\left(\partial^\mu\Phi\partial^\nu\Phi\nabla_\mu\nabla_\nu - \partial_\mu\Phi\partial^\mu\Phi\nabla^\rho\nabla_\rho\right)^n\right.$$
$$\left.\left.\times\mathcal{P}^{\alpha\mu}\mathcal{P}^{\beta\nu}\left(R_{\mu\nu} - \frac{1}{2\bar{W}}\partial_\rho\Phi\nabla^\rho\nabla_\mu\nabla_\nu\Phi\right)\right] - \lambda_{\rm L}\left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \bar{W}\right)\right\}\,, \quad (24)$$

$$S_{2n+3} = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \zeta\left[\left(\partial^\mu\Phi\partial^\nu\Phi\nabla_\mu\nabla_\nu - \partial_\mu\Phi\partial^\mu\Phi\nabla^\rho\nabla_\rho\right)^n {\mathcal{P}_\alpha}^\mu {\mathcal{P}_\beta}^\nu\right.\right.$$
$$\left.\times\left(R_{\mu\nu} - \frac{1}{2\bar{W}}\partial_\rho\Phi\nabla^\rho\nabla_\mu\nabla_\nu\Phi\right)\right]\left[\left(\partial^\mu\Phi\partial^\nu\Phi\nabla_\mu\nabla_\nu - \partial_\mu\Phi\partial^\mu\Phi\nabla^\rho\nabla_\rho\right)^{n+1}\right.$$
$$\left.\left.\times\mathcal{P}^{\alpha\mu}\mathcal{P}^{\beta\nu}\left(R_{\mu\nu} - \frac{1}{2\bar{W}}\partial_\rho\Phi\nabla^\rho\nabla_\mu\nabla_\nu\Phi\right)\right] - \lambda_{\rm L}\left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \bar{W}\right)\right\}\,. \quad (25)$$

Here, Eqs. (24) and (25) are for $z = 2n+2$ and $z = 2n+3$ $(n = 0, 1, 2, \dots)$, respectively, where z is the quantity denoting the anisotropy between the time and spatial coordinates [30]. Moreover, the gravitational field equation is given by $\left[1/\left(2\kappa^2\right)\right]\left[R_{\mu\nu} - (1/2)\, g_{\mu\nu} R\right] + G_{\mu\nu}^{(\mathrm{higher})} - (\lambda_{\mathrm{L}}/2)\, \partial_\mu \Phi \partial_\nu \Phi + (1/2)\, g_{\mu\nu} \left[(1/2)\, \partial_\rho \Phi \partial^\rho \Phi + \bar{W}\right] = 0$ with $G_{\mu\nu}^{(\mathrm{higher})}$ the higher derivative term, i.e., the second term, in the actions in Eq. (24) and (25). Suppose the flat vacuum solution, the constraint equation $(1/2)\, \partial_\mu \Phi \partial^\mu \Phi + \bar{W} = 0$ becomes $(1/2)\, (d\Phi/dt)^2 = \bar{W}$. For the flat space solution, the gravitational field equation is reduced to $\lambda_{\mathrm{L}} \partial_\mu \Phi \partial_\nu \Phi = 0$, because all of the term $\nabla_\mu \nabla_\nu \Phi$ as well as the curvature terms vanish. The solution is given by $\lambda_{\mathrm{L}} = 0$, because $\partial_\mu \Phi \neq 0$ owing to $(1/2)\, \partial_\mu \Phi \partial^\mu \Phi + \bar{W} = 0$. Thus, in these actions in Eqs. (24) and (25) solutions with $\lambda_{\mathrm{L}} = 0$ in the flat space vacuum can be realized. We further analyze the perturbations $h_{\mu\nu}$ with $\lambda_{\mathrm{L}} = 0$. Using the diffeomorphism invariance in terms of the time coordinate, as a gauge condition for the unitarity, we set $\Phi = \sqrt{2\bar{W}}t$. By taking only the quadratic terms of the perturbations, we rewrite the actions in Eqs. (24) and (25). At this stage, there remains the diffeomorphism invariance in terms of the spatial coordinates. In addition, the term h_{0i} in the higher derivative term with a coefficient ζ does not exist in the rewritten actions. Moreover, the above constraint equation leads to $h_{00} = 0$. We derive the equations by varying the actions in Eqs. (24) and (25) with respect to h_{00} and Ψ. We decompose h_{0i} as $h_{0i} \partial_i s + v_i$ with $\partial^i v_i = 0$, where s is the spatial scalar quantity and v_i is a vector field. The invariance in terms of the spatial coordinates under the transformations of the linearized diffeomorphism is described as $\delta x^i = \partial^i u + w^i$ with $\partial_i w^i = 0$, where u is the spatial scalar quantity and w_i is a vector field. We find the following transformations under the diffeomorphism: $\delta s = \partial_t u$ and $\delta v_i = \partial_t w_i$. Accordingly, the gauge condition $s = v^i = 0$, i.e., $h_{ti} = 0$ can be chosen. Furthermore, we express h_{ij} as $h_{ij} = \delta_{ij} A + \partial_j B_i + \partial_i B_j + C_{ij} + \left[\partial_i \partial_j - (1/3)\, \delta_{ij} \partial_k \partial^k\right] E$, where A and E are scalar quantities, B_i is a vector field, and C_{ij} is a tensor field. Here, $\partial^i B_i = 0$, $\partial^i C_{ij} = \partial^j C_{ij} = 0$, and $C_i^{\ i} = 0$. As a result, we acquire $\lambda_{\mathrm{L}} = 0$ and vector $B_i = 0$, and thus the scalar λ_{L} and vector B_i modes do not propagate. We fix the gauge in the actions rewritten above. We vary these actions over A and E and obtain equations. From these equations and $A = (1/3)\, \partial_k \partial^k E$, which is derived by the equation derived by the variation of the action with respect to h_{0i} with the above decomposed representation of h_{ij}, we find $\partial_0^2 A = 0$. Hence, since A and E have only the dependence on the spatial coordinate, these scalar quantities do not propagate. As a consequence, all the scalar modes Φ, λ_{L}, h_{00}, s, A and E and all the vector modes v_i and B_i do not propagate, whereas the propagating mode is only the tensor mode C_{ij}, namely, a massless graviton. This is a different feature from the Hořava gravity [30] without the Lorentz invariance. The final expressions of the actions in Eqs. (24) and (25) are given by

$$S_{2n+2} = \int d^4x \left\{ \frac{1}{8\kappa^2} \left[C_{ij} \left(-\partial_0^2 + \partial_k \partial^k \right) C^{ij} \right] - 2^{2n-2} \zeta \bar{W}^{2n} \left[\left(\partial_k \partial^k \right)^{n+1} C_{ij} \right] \left[\left(\partial_k \partial^k \right)^{n+1} C^{ij} \right] \right\} , \quad (26)$$

$$S_{2n+3} = \int d^4x \left\{ \frac{1}{8\kappa^2} \left[C_{ij} \left(-\partial_0^2 + \partial_k \partial^k \right) C^{ij} \right] - 2^{2n-1} \zeta \bar{W}^{2n+1} \left[\left(\partial_k \partial^k \right)^{n+1} C_{ij} \right] \left[\left(\partial_k \partial^k \right)^{n+2} C^{ij} \right\} \right] . \quad (27)$$

Therefore, in the momentum space the propagator reads

$$\begin{aligned} &\langle h_{ij}(p) h_{kl}(-p) \rangle = \langle C_{ij}(p) C_{kl}(-p) \rangle \\ &= \frac{1}{2} \left[\left(\delta_{ij} - \frac{p_i p_j}{\boldsymbol{p}^2} \right) \left(\delta_{kl} - \frac{p_k p_l}{\boldsymbol{p}^2} \right) - \left(\delta_{ik} - \frac{p_i p_k}{\boldsymbol{p}^2} \right) \left(\delta_{jl} - \frac{p_j p_l}{\boldsymbol{p}^2} \right) \right. \\ &\left. - \left(\delta_{il} - \frac{p_i p_l}{\boldsymbol{p}^2} \right) \left(\delta_{jk} - \frac{p_j p_k}{\boldsymbol{p}^2} \right) \right] \\ &\times \begin{cases} 1/\left[p^2 - 2^{2n} \zeta \kappa^2 \bar{W}^{2n} \boldsymbol{p}^{4(n+1)} \right] , & \text{for} \quad z = 2n+2 \\ 1/\left[p^2 - 2^{2n-1} \zeta \kappa^2 \bar{W}^{2n+1} \boldsymbol{p}^{2(2n+3)} \right] , & \text{for} \quad z = 2n+3 \end{cases} , \end{aligned} \quad (28)$$

with $\boldsymbol{p}^2 = \sum_{i=1}^{3} \left(p^i \right)^2$ and $p^2 = -\left(p^0 \right)^2 + \boldsymbol{p}^2$. If $\zeta > 0$ and $p^0 = 0$, when $2^{2n} \zeta \kappa^2 \bar{W}^{2n} \boldsymbol{p}^{4n+2} = 1$ for $z = 2n+2$ and $2^{2n-1} \zeta \kappa^2 \bar{W}^{2n+1} \boldsymbol{p}^{4(n+1)} = 1$ for $z = 2n+3$ are satisfied, the tachyonic pole exists. Thus, at least the flat vacuum is unstable. In this model, at least on the tree level, no propagating vector or scalar mode exists. The fact that the tensor structure of the propagator in Eq. (28) changes implies that the vector or scalar mode could emerge. In other words, the vector or scalar mode has to be a composite state. At any perturbative level, this does not appear usually. Accordingly, the quantum corrections should not change the tensor structure. In the UV region with large $\boldsymbol{k}$, for $z = 2$ $(n = 0)$ in Eq. (26), the propagator evolves as $1/\left|\boldsymbol{k}\right|^4$. Hence, the UV behavior performs. While, for $z = 3$ $(n = 0)$ in Eq. (27), the propagator evolves as $1/\left|\boldsymbol{k}\right|^6$. Thus, the model is power-counting renormalizable. For $z = 2n+2$ $(n \geqslant 1)$ in Eq. (26) or $z = 2n+3$ $(n \geqslant 1)$ in Eq. (27), the model is power-counting super-renormalizable. In the high energy regime, the dispersion relation of the graviton becomes $\omega = \bar{c} k^z$ with $\bar{c} (> 0)$ a positive constant for the consistency of the dispersion relation. Here, ω is the angular frequency which corresponds to the energy and k is the wave number which does to the momentum. Accelerating cosmology in such a theory was studied in Ref. [6].

6. Conclusions

We have studied the accelerating (dark energy) solutions of modified gravity. These solutions may yield future singularities. We have explored the finite-time future singularities in $F(R)$, $F(G)$ and $\mathcal{F}(R,G)$ gravity theories. The removal of the finite-time future singularities in $F(R)$ gravity by adding an R^2-term which simultaneously leads to the unification of early-time inflation with late-time acceleration [9] has been mentioned. The corresponding term may be different for $F(G)$ or $\mathcal{F}(R,G)$ gravity theory [3]. Moreover, we have studied dark energy in $F(R)$ gravity with the Lagrange multiplier field. Furthermore, domain wall solutions in $F(R)$ gravity have been presented. In addition, we have investigated covariant higher derivative gravity with scalar projectors.

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