Some representations of planar Galilean conformal algebra*

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Abstract

Representation theory of an infinite dimensional Galilean conformal algebra introduced by Martelli and Tachikawa is developed. We focus on the algebra defined in $(2+1)$ dimensional spacetime and consider central extension. It is then shown that the Verma modules are irreducible for non-vanishing highest weights. This is done by explicit computation of Kac determinant. We also present coadjoint representations of the Galilean conformal algebra and its Lie group. As an application of them, a coadjoint orbit of the Galilean conformal group is given in a simple case.

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1. Introduction

Representation theories of infinite dimensional Lie algebras such as Virasoro and Kac-Moody are, as is widely known, fruitful field of mathematical and physical research. Those Lie algebras are associated with certain semisimple finite dimensional Lie algebras. Recent renewed interest in conformal algebras in nor-relativistic setting, especially in the context of non-relativistic AdS/CFT correspondence [1, 2], introduced a new class of infinite dimensional Lie algebras associated with certain non-semisimple Lie algebras [2, 3, 4]. The best studied example of this class of algebra is the Schrödinger-Virasoro algebra [3] associated with six dimensional Schrödinger algebra. Readers may refer the book [5] for background, definition and representation theory of Schrödinger-Virasoro algebra.

In the present work we investigate representations of an another algebra of this class introduced in [2]. In [2] a two-parameter family of infinite

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dimensional algebra is discussed. One of the parameters is the dimension of space d and the other is "spin" ℓ. Here we pick up the algebra corresponding to $d = 2,\ \ell = 1$ and denote it by $\mathfrak{g}_0$. The basis of $\mathfrak{g}_0$ is L_m, P^i_n and J_m with $m \in \mathbb{Z},\ i, j = 1, 2$. The defining relations of the algebra are given by

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n}, \\ &[L_m, P^i_n] = (m-n)P^i_{m+n}, \qquad [L_m, J_n] = -nJ_{m+n}, \\ &[J_m, P^i_n] = \sum_j \epsilon_{ij} P^j_{m+n}, \qquad [J_m, J_n] = [P^i_m, P^j_n] = 0, \end{aligned} \tag{1}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. It is seen that $\langle L_m \rangle$ form a centerless Virasoro algebra and $\langle P^i_n \rangle, \langle J_n \rangle$ carry a representation of it. Moreover, $\langle P^i_n \rangle$ is an Abelian ideal of $\mathfrak{g}_0$. Following vector field realization of $\mathfrak{g}_0$ is helpful to see the algebra is indeed related to $(2+1)$ dimensional spacetime:

$$\begin{aligned} &L_m = -t^{m+1}\partial_t - (m+1)t^m x_i \partial_{x_i}, \\ &J_m = -t^m(x_1\partial_{x_2} - x_2\partial_{x_1}), \qquad P^i_m = -t^{m+1}\partial_{x_i}. \end{aligned} \tag{2}$$

The subspace spanned by $\langle L_0, L_{\pm 1}, J_0, P^i_0, P^i_{\pm 1} \rangle$ forms a ten dimensional subalgebra also called Galilean conformal algebra. It is known that this subalgebra has a peculiar central extension in the sense that it exists for only $d = 2$ and integral values of ℓ [2, 6]. This *exotic* central extension makes the Abelian ideal $\langle P^i_m \rangle$ makes non-Abelian:

$$[P^i_m, P^j_n] = I_{mn}\epsilon^{ij}\Theta,$$

where I_{mn} is a symmetric tensor. It is also known that the exotic Galilean conformal algebra has some physical applications [2, 6]. (For more details on finite dimensional Galilean conformal algebras, see [7] and references therein). In fact, these observation on the finite dimensional exotic Galilean conformal algebra is a motivation of the present work. The algebra $\mathfrak{g}_0$ is an infinite dimensional version of the Galilean conformal algebra for $d = 2,\ \ell = 1$. Then natural questions arise. Dose $\mathfrak{g}_0$ have a central extension of exotic type ? In what kind of physical context $\mathfrak{g}_0$ appears ? We shall give a negative answer to the first question: $\mathfrak{g}_0$ does not have the exotic central extension but others. We have no idea on the second question at this writing. However, we know that physical application of algebraic object is available via its representations. Therefore we study representations of $\mathfrak{g}_0$ with central extensions.

The plan of this paper is as follows: In the next section, we introduce central extensions of $\mathfrak{g}_0$ and denote the algebra with the central extensions by $\mathfrak{g}$. We then consider highest weight representations of $\mathfrak{g}$, especially we investigate irreducibility of Verma modules over $\mathfrak{g}$. This is done by calculating Kac determinant. We give a explicit formula of the Kac determinant and it shows that the Verma modules are irreducible for non-vanishing highest weights. In §3 we investigate coadjoint representations of $\mathfrak{g}$. By employing the *regular dual* of $\mathfrak{g}$ as a space $\mathfrak{g}^*$ dual to $\mathfrak{g}$, we calculate the action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This allows us to compute coadjoint orbit of the infinite dimensional group $\mathfrak{G}$ having $\mathfrak{g}$ as its Lie algebra. We shall give a simple example of coadjoint orbit. However, the coadjoint action of $\mathfrak{g}$ is not enough to classify all coadjoint orbits of $\mathfrak{G}$ [5]. We need coadjoint representation of the group $\mathfrak{G}$. This is presented in §4.

2. Verma modules and its irreducibility

We start with central extensions of $\mathfrak{g}_0$. Our interest is in the possibility of the exotic central extension which makes Abelian ideal non-Abelian. It is also curious whether the Virasoro subalgebra has the ordinary central extension. We here present only the result and the proof is found in [8].

Proposition 1 *The algebra $\mathfrak{g}_0$ has the following central extensions*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12}m(m^2-1)\delta_{m+n,0}, \tag{3}$$

$$[J_m, J_n] = \beta m \delta_{m+n,0}, \tag{4}$$

where α, β are central charges. However, the exotic type central extension is impossible.

We denote the algebra with the central extensions by $\mathfrak{g}$. In the rest of this paper we investigate representation of $\mathfrak{g}$.

Our main subject in this section is highest weight representations of $\mathfrak{g}$, especially we consider Verma modules. Important fact is that the algebra $\mathfrak{g}$ admits the triangular decomposition. Define the degree of $X_n \in \mathfrak{g}$ by $\deg(X_n) = -n$ where $X = L, J, P^i$. This allows us to define the triangular decomposition of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+ \\ &= \langle\, L_{-n}, J_{-n}, P^i_{-n} \,\rangle \oplus \langle\, L_0, J_0, P^i_0 \,\rangle \oplus \langle\, L_n, J_n, P^i_n \,\rangle, \quad n \in \mathbb{N} \end{aligned} \tag{5}$$

Let $|0\rangle$ be the highest weight vector:

$$\mathfrak{g}^+ |0\rangle = 0, \quad L_0 |0\rangle = h |0\rangle, \quad J_0 |0\rangle = \mu |0\rangle, \quad P^i_0 |0\rangle = \rho_i |0\rangle. \tag{6}$$

Following the usual definition of Verma modules, we define the Verma modules over $\mathfrak{g}$ by

$$V^\chi = U(\mathfrak{g}^-) \otimes |0\rangle, \qquad \chi = (h, \mu, \rho_1, \rho_2, \alpha, \beta), \tag{7}$$

where $U(\mathfrak{g}^-)$ denotes the universal enveloping algebra of $\mathfrak{g}^-$. The Verma module V^χ is a graded-modules through a natural extension of the degree from $\mathfrak{g}$ to $U(\mathfrak{g})$ by $\deg(XY) = \deg(X) + \deg(Y)$, $X, Y \in U(\mathfrak{g})$,

$$V^\chi = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^\chi_n, \quad V^\chi_n = \{ X |0\rangle \mid X \in U(\mathfrak{g}^-),\ \deg(X) = n \}.$$

There exists an algebraic anti-automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\omega(L_m) = L_{-m}, \qquad \omega(J_m) = J_{-m}, \qquad \omega(P^i_m) = P^i_{-m}. \tag{8}$$

One can introduce an inner product in V^χ by extending the anti-automorphism ω to $U(\mathfrak{g})$. We define the inner product of $X\,|0\rangle\,, Y\,|0\rangle \in V^\chi$ by

$$\langle 0|\, \omega(X) Y\,|0\rangle\,, \qquad \langle 0|0\rangle = 1.$$

The reducibility of V^χ may be investigated by the Kac determinant. The Kac determinant is defined as usual [9]. Let $|i\rangle\,(i = 1, \cdots, \dim V_n^\chi)$ be a basis of V_n^χ, then the Kac determinant at level (degree) n is given by

$$\Delta_n = \det(\ \langle i|j\rangle\ ).$$

We want to calculate Δ_n. The main obstacles of this calculation are rapid increase of dim V_n^χ as a function of n and Δ_n is never reduced to the determinant of diagonal matrix. For illustration we list $\dim V_n^\chi$ for some small n:

n	0	1	2	3	4	5
$\dim V_n^\chi$	1	4	14	40	105	252

However, one can carry out the computation by the method similar to that for Schrödinger-Virasoro algebra used in [5]. We here merely mention the result and do not go into the detail. The computational details are found in [8]. To mention the result we need some preparation.

A partition $A = (a_1 a_2 \cdots a_\ell)$ of a positive integer a is the sequence of positive integers such that

$$\begin{aligned} a &= a_1 + a_2 + \cdots + a_\ell, \\ & a_1 \geq a_2 \geq \cdots \geq a_\ell > 0. \end{aligned}$$

The integers ℓ is called length of the partition A and denoted by $\ell(A)$. For a given partition A of a, we decompose a set of integers $a_1, a_2, \cdots, a_\ell$ to two subsets by selecting s integers from them $(0 \leq s \leq \ell)$:

$$A_1 = \{\ a_{\sigma_1} \geq a_{\sigma_2} \geq \cdots \geq a_{\sigma_s}\ \}, \qquad A_2 = \{\ a_{\rho_1} \geq a_{\rho_2} \geq \cdots \geq a_{\rho_{\ell-s}}\ \}. \tag{9}$$

A_1 consists of the selected s integers and the members of A_2 are the rest integers so that the partition A is decomposed into a pair of partitions $(A_1 A_2)$. We denote the number of all possible pairs $(A_1 A_2)$ by $s(A)$. For instance, let $A = (21)$ then the possible pairs are $(A_1 A_2) = ((21)\phi),\ ((2)(1)),\ ((1)(2)),\ (\phi(21))$ so that $s((21)) = 4$. Now we mention our result of Δ_n :

Theorem 1 *Level n Kac determinant is given by*

$$\Delta_n = c_n \prod_{a,b} \prod_{A,B} (\rho_1^2 + \rho_2^2)^{\frac{1}{2} s(A) s(B) (\ell(A) + \ell(B))}$$

where the pair (a, b) runs all possible non-negative integers satisfying $n = a + b$ and the pair (A, B) runs all possible partitions of fixed a and b. The coefficient c_n is a numerical constant.

Explicit values of c_n (up to sign) and the power of $\rho_1^2+\rho_2^2$ for $n=1,2,3$ are listed below:

n	1	2	3
(c_n, power)	$(2,2)$	$(2^{18},12)$	$(2^{72}3^6,48)$

We remark that Δ_n is independent of the central charges α,β. Thus the formula of Δ_n is common for the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$. We see from Theorem 1 that if $\rho_1^2+\rho_2^2\neq 0$ then the Kac determinant Δ_n never vanish so that there exist no singular vectors in V_n^χ for any n.

Corollary 1 *The Verma module V^χ is irreducible if $\rho_1^2+\rho_2^2\neq 0$.*

This is a sharp contrast to the corresponding finite dimensional algebra with the exotic central extension where some Verma modules for certain non-vanishing highest weights are reducible [10]. However, similar irreducibility of Verma modules for nonvanishing highest weights is also observed for the Schrödinger-Virasoro algebra [5].

3. Coadjoint representation of $\mathfrak{g}$ and coadjoint orbit

Our next subject is coadjoint representations. Physical implication of coadjoint representations is laid in coadjoint orbits of the infinite dimensional Lie *group* $\mathfrak{G}$ which is an integration of $\mathfrak{g}$. As is known widely, any coadjoint orbit is a symplectic manifold with Poisson structure so that the base of geometric quantization [11]. We thus want to derive coadjoint representation of $\mathfrak{G}$ and make a classification of coadjoint orbits. However, this work is in progress. Here we shall give a simple example of the coadjoint orbit using a coadjoint representation of $\mathfrak{g}$ which is calculated by employing the method in [5].

Coadjoint representations are, by definition, obtained by the action of $\mathfrak{g}$ or $\mathfrak{G}$ on the space $\mathfrak{g}^*$ dual to $\mathfrak{g}$. Thus we have to determine the space $\mathfrak{g}^*$. To this end, we return to the centerless algebra $\mathfrak{g}_0$ and recast it in the form of the current:

$$L_f=f(\theta)\partial_\theta+f'(\theta)x_i\partial_{x_i},\quad J_f=f(\theta)(x_1\partial_{x_2}-x_2\partial_{x_1}),\quad P_f^i=f(\theta)\partial_{x_i},$$

where $\theta\in S^1$ is a compactified time coordinate. Then we have a current algebra with the commutation relations:

$$\begin{aligned}[][L_f,L_g]=L_{fg'-f'g},\quad [L_f,P_g^i]=P^i_{fg'-f'g},\quad [L_f,J_g]=J_{fg'},\\ [J_f,P_g^i]=-\sum_k \epsilon_{ik}P^k_{fg},\quad [J_f,J_g]=[P_f^i,P_g^j]=0.\end{aligned}\tag{10}$$

By considering Fourier components one may recover the relations (1) and (2). Now let us recall that the centerless Virasoro algebra has the one-parameter family of representation on the space of *densities* $\mathcal{F}_\lambda$ of the form $\phi(\theta)d\theta^{-\lambda}$ [5] (see also [9]). The action of L_f on $\mathcal{F}_\lambda$ is defined by

$$L_f(\ \phi(\theta)d\theta^{-\lambda}\ ):=(f\phi'-\lambda f'\phi)d\theta^{-\lambda}.\tag{11}$$

The dual space $\mathcal{F}_\lambda^*$ may be identified with $\mathcal{F}_{-1-\lambda}$ through the paring:

$$\begin{aligned} &\mathcal{F}_\lambda^* \times \mathcal{F}_\lambda \to \mathbb{C}, \\ &\langle\ u(\theta) d\theta^{1+\lambda},\ f(\theta)\theta d\theta^{-\lambda}\ \rangle = \int_{S^1} u(\theta) f(\theta) d\theta. \end{aligned} \tag{12}$$

By comparing (10) and (11) we are allow to make the following identification:

$$L_g \simeq \mathcal{F}_1, \quad P_g^i \simeq \mathcal{F}_1, \quad J_g \simeq \mathcal{F}_0.$$

Thus $\mathfrak{g}_0 \simeq \mathcal{F}_1 \oplus \mathcal{F}_1 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$ as a vector space. It follows that

$$\mathfrak{g}_0^* \simeq \mathcal{F}_{-2} \oplus \mathcal{F}_{-2} \oplus \mathcal{F}_{-2} \oplus \mathcal{F}_{-1}$$

Next we consider the central extensions in Proposition 1. The algebra $\mathfrak{g}$ may be regarded as

$$\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathbb{R} \oplus \mathbb{R}.$$

Therefore, we obtain the algebra dual to $\mathfrak{g}$ as follows:

$$\mathfrak{g}^* \simeq \mathcal{F}_{-2} \oplus \mathcal{F}_{-2} \oplus \mathcal{F}_{-2} \oplus \mathcal{F}_{-1} \oplus \mathbb{R} \oplus \mathbb{R}. \tag{13}$$

We denote an element of $\mathfrak{g}^*$ by

$$\gamma_0 d\theta^2 + \gamma_1 d\theta^2 + \gamma_2 d\theta^2 + \gamma_3 d\theta + a + b \ \in\ \mathfrak{g}^* \tag{14}$$

and identify it with the column vector $\vec{\gamma} = {}^t(\gamma_0, \gamma_1, \gamma_2, \gamma_3, a, b)$. Then the duality pairing is given by

$$\langle \vec{\gamma}, \vec{X} \rangle = \sum_{i=0}^{3} \int_{S^1} \gamma_i f_i \, d\theta + a\alpha + b\beta, \tag{15}$$

where $\vec{X} = (L_{f_0}, P_{f_1}^1, P_{f_2}^2, J_{f_3}, \alpha, \beta)$.

We now come to the definition of coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Define an action $X(\gamma)$ of $\mathfrak{g}$ on $\mathfrak{g}^*$ by

$$\langle X(\gamma), Y \rangle := -\langle \gamma, [X, Y] \rangle, \quad X, Y \in \mathfrak{g},\ \gamma \in \mathfrak{g}^*, \tag{16}$$

then $X(\gamma)$ gives a representation of $\mathfrak{g}$ [11]. It is not difficult to calculate the coadjoint action according to the definition. The result is summarized as follows.

Proposition 2 *Coadjoint representation of* $\mathfrak{g}$ *is given by*

$$\begin{aligned} &L_{f_0}\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix} a f_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 \\ 2\gamma_1 f_0' + \gamma_1' f_0 \\ 2\gamma_2 f_0' + \gamma_2' f_0 \\ \gamma_3 f_0' + \gamma_3' f_0 \end{pmatrix}, \quad J_{f_3}(\vec{\gamma}) = \begin{pmatrix} \gamma_3 f_3' \\ \gamma_2 f_3 \\ -\gamma_1 f_3 \\ b f_3' \end{pmatrix}, \\ &P_{f_1}^1(\vec{\gamma}) = \begin{pmatrix} 2\gamma_1 f_1' + \gamma_1' f_1 \\ 0 \\ 0 \\ -\gamma_2 f_1 \end{pmatrix}, \qquad P_{f_2}^2(\vec{\gamma}) = \begin{pmatrix} 2\gamma_2 f_2' + \gamma_2' f_2 \\ 0 \\ 0 \\ -\gamma_1 f_2 \end{pmatrix}. \end{aligned} \tag{17}$$

The coadjoint action on the central elements a, b is trivial so that they are omitted.

We remark that $L_{f_0}(\gamma_0)$ is identical to the coadjoint action of the Virasoro algebra [12, 13].

Using the formulae in Proposition 2 one may calculate coadjoint orbits. By definition, coadjoint orbit is an orbit of the group $\mathfrak{G}$ in the space $\mathfrak{g}^*$. To find the orbits we have to find an isotropy group $\mathfrak{H} \subset \mathfrak{G}$ for a given $\vec{\gamma} \in \mathfrak{g}^*$. Then one may identify the orbit with $\mathfrak{G}/\mathfrak{H}$. Here we consider an infinitesimal transformation of $\vec{\gamma}$ by $\mathfrak{G}$, namely, we look for an isotropy algebra of $\vec{\gamma}$. To this end, one has to solve the equation:

$$(L_{f_0} + J_{f_3} - P^1_{f_1} - P^2_{f_2})(\vec{\gamma}) = 0. \tag{18}$$

We show only a simple example of the solution of the equation (18). Full classification of the coadjoint orbits of the group $\mathfrak{G}$ will be presented elsewhere. We assume that γ_1 and γ_2 are nonvanishing constants. Then the equation (18) is reduced to a set of four equations.

$$\begin{aligned}
&af_0''' + 2\gamma_0 f_0' + \gamma_0' f_0 + \gamma_3 f_3' = 2(\gamma_1 f_1' + \gamma_2 f_2'),\\
&2\gamma_1 f_0' + \gamma_2 f_3 = 0,\\
&2\gamma_2 f_0' - \gamma_1 f_3 = 0,\\
&\gamma_3 f_0' + \gamma_3' f_0 + b f_3' = -\gamma_2 f_1 + \gamma_1 f_2.
\end{aligned}$$

This set of equation is easily solved to give

$$f_0 = const., \quad f_3 = 0, \quad f_k = \Phi_k(\theta) f_0 + const, \ k = 1, 2 \tag{19}$$

where $\Phi_k(\theta)$ is a functions of $\gamma_0(\theta), \ \gamma_3(\theta)$. We thus have the following generators of the isotropy algebra:

$$L_{f_0} = const. \partial_\theta, \qquad P^k_{f_k} = f_k(\theta)\partial_{x_k}, \qquad J_{f_3} = 0. \tag{20}$$

One may see that L_{f_0} generate a translation on S^1 and $f_k(\theta)$ with $k = 1, 2$ is an arbitrary function. It follows that the isotropy group for γ_1, γ_2 being nonvanishing constant is a semidirect product of $U(1)$ by the group $\mathfrak{G}_P$ generated by $P^k_{f_k}$. Recalling that the group $\mathfrak{G}$ is a semidirect product of the Virasoro group $\mathrm{Diff}(S^1)$ by $\mathfrak{G}_P$ and $\widehat{SO(2)}$, we conclude that the coadjoint orbit for γ_1, γ_2 being non-vanishing constant is a product of a generic orbit $\mathrm{Diff}(S^1)$ of the Virasoro group by $\widehat{SO(2)}$.

4. Coadjoint representation of $\mathfrak{G}$

In this section we give the group $\mathfrak{G}$ more explicitly and calculate its coadjoint action which will be used to determine all coadjoint orbits of $\mathfrak{G}$. The Lie algebra $\mathfrak{g}$ is a semidirect sum of the Virasoro algebra and the algebra

$\langle P^k_{f_k}, J_{f_3}\rangle$. Since integration of the Virasoro algebra is well-known, we first consider the group generated by $P^k_{\eta_k}$ and J_ξ. We denote an element of the group by (ξ, η_1, η_2). Regarding $\exp(J_\xi)\exp(P^1_{\eta_1} + P^2_{\eta_2})$ as (ξ, η_1, η_2), one may deduce the following formulae for group multiplication and inverse:

$$\begin{aligned}
&(\xi, \eta_1, \eta_2)(\rho, \sigma_1, \sigma_2) = \\
&\qquad (\xi + \rho, \sigma_1 + \eta_1 \cos\rho - \eta_2 \sin\rho, \sigma_2 + \eta_1 \sin\rho + \eta_2 \cos\rho)\exp(c\beta), \\
&(\xi, \eta_1, \eta_2)^{-1} = (-\xi, -\eta_1 \cos\xi - \eta_2 \sin\xi, \eta_1 \sin\xi - \eta_2 \cos\xi).
\end{aligned}$$

The group $\mathfrak{G}$ is defined by

$$(\phi, \xi, \eta_1, \eta_2) := (id, \xi, \eta_1, \eta_2)(\phi, 0, 0, 0), \qquad \phi \in \mathrm{Diff}(S^1) \tag{21}$$

It is not verify the relation

$$(\phi, 0, 0, 0)(id, \xi, \eta_1, \eta_2) = \left(\phi, \xi \circ \phi, \frac{\eta_1 \circ \phi}{\phi'}, \frac{\eta_2 \circ \phi}{\phi'}\right). \tag{22}$$

Coadjoint action of $\mathfrak{G}$ on $\mathfrak{g}^*$ is defined by [11]

$$\langle g(\gamma), X\rangle := \langle \gamma, g^{-1}Xg\rangle, \quad X \in \mathfrak{g}, \quad \gamma \in \mathfrak{g}^*, \quad g \in \mathfrak{G} \tag{23}$$

This gives a representation of $\mathfrak{G}$. Now we are ready to calculate coadjoint action of $\mathfrak{G}$. We treat the Virasoro subgroup separately.

Proposition 3 *Coadjoint action of* $\mathrm{Diff}(S^1)$ *on* $\mathfrak{g}^*$ *is given by*

$$\phi(\vec{\gamma}) = \begin{pmatrix} a\Theta(\phi) + (\gamma_0 \circ \phi)(\phi')^2 \\ (\gamma_1 \circ \phi)(\phi')^2 \\ (\gamma_2 \circ \phi)(\phi')^2 \\ (\gamma_3 \circ \phi)\phi' \end{pmatrix}, \quad \Theta(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2,$$

where $\Theta(\phi)$ *denotes the Schwarzian derivative of* θ*. Action on the central elements is omitted.*

One may easily prove Proposition 2. $\phi(\gamma_0)$ is nothing but the well-known coadjoint action of the Virasoro group [12, 13]. Denoting the coordinate change on S^1 due to $\mathrm{Diff}(S^1)$ by $\theta = \phi(t)$, we have

$$\gamma(\theta)d\theta^\lambda = \gamma(\phi(t))\left(\frac{d\theta}{dt}dt\right)^\lambda = (\gamma \circ \phi)(\phi')^\lambda dt^\lambda.$$

$\phi(\gamma_k)$ for $k = 1, 2, 3$ is immediately obtained from this relation.

Let us turn to coadjoint action of the (ξ, η_1, η_2)-subgroup. It is calculated straightforwardly by using definition (23). As an illustration we compute $(\xi, \eta_1, \eta_2)(\gamma_1)$. Let $g = (\xi, \eta_1, \eta_2)$, then it is easy to see that

$$g^{-1} P^1_{f_1} g = P^1_{f_1} \ \cos\xi + P^2_{f_2} \ \sin\xi.$$

By (23)

$$\langle g(\vec{\gamma}), P^1_{f_1} \rangle = \langle \vec{\gamma}, g^{-1} P^1_{f_1} g \rangle = \int_{S^1} (\gamma_1 \cos\xi + \gamma_2 \sin\xi) f_1 d\theta.$$

It follows that $g(\gamma_1) = \gamma_1 \cos\xi + \gamma_2 \sin\xi$. $g(\gamma_k)$ for $k = 0, 2, 3$ is computed in a similar way and result is summarized as follows:

Proposition 4 *Coadjoint representation of (ξ, η_1, η_2)-subgroup is given by*

$$(\xi, \eta_1, \eta_2)(\vec{\gamma}) = \begin{pmatrix} \gamma_0 + \sum_{k=1,2} (\gamma_k' \eta_k + 2\gamma_k \eta_k') + (\gamma_1 \eta_2 - \gamma_2 \eta_1 + \gamma_3) \xi' + \frac{b}{2} (\xi')^3 \\ \gamma_1 \cos\xi + \gamma_2 \sin\xi \\ -\gamma_1 \sin\xi + \gamma_2 \cos\xi \\ \gamma_1 \beta_2 - \gamma_2 \beta_1 + \gamma_3 + b\xi' \end{pmatrix}.$$

Action on the central elements is omitted.

5. Concluding remarks

We studied infinite dimensional Galilean conformal algebra introduced by Martelli and Tachikawa. We concentrated on the algebra defined in $(2+1)$ dimensional spacetime and took into account the central extensions. Our main results are the proof of irreducibility of the Verma modules over $\mathfrak{g}$ and explicit formula of the coadjoint representations of $\mathfrak{g}$ and $\mathfrak{G}$. We also presented a coadjoint orbit of $\mathfrak{G}$ for a simple case.

The present work will be followed by a full classification of coadjoint orbit of $\mathfrak{G}$. This will be done by using Proposition 3 and 4. Another interesting problem in representation theory is action of $\mathfrak{G}$ on a space of differential operators. It is known that Virasoro group acts on the space of Hill operator and Hill operator gives a Lax form of KdV equation [13]. It is also known that Schrödinger-Virasoro algebra acts on the space of Schrödinger operator [5]. Is it possible to represent $\mathfrak{G}$ on certain space of differential operator ? If so, what is the space ? Answer to this question would open a way to the most interesting problem on $\mathfrak{g}$ or $\mathfrak{G}$, namely, physical applications.

We focus on the algebra in $(2+1)$ dimensional spacetime, however, algebra $\mathfrak{g}_0$ is also defined for other dimensional spacetime. Furthermore, as mentioned in §1, there are some variety of infinite dimensional Galilean conformal algebra [4]. Nevertheless, only limited number of works has been published on those algebraic structure. They are waiting for extensive study from physical and mathematical points of view.

References

[1] D. T. Son, *Phys. Rev.* **D 78** (2008) 046003; K. Balasubramanian and J. McGreevy, *Phys. Rev. Lett.* **101** (2008) 061601.

[2] D. Martelli and Y. Tachikawa, *J. High Energy Phys.* **05** (2010) 091.

[3] M. Henkel, *J. Stat. Phys.* **75** (1994) 1023.

[4] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, *J. High Energy Phys.* **08** (2010) 004; M. Alishahiha, A. Davody and A. Vahedi, *J. High Energy Phys.* **08** (2009) 022; A. Hosseiny and S. Rouhani, *J. Math. Phys.* **51** (2010) 102303; *ibid.* 052307; C. Duval and P. A. Horváthy, *J. Phys.* **A**:*Math. Theor.* **42** (2009) 465206.

[5] C. Roger and J. Unterberger, *The Schrödinger-Virasoro Algebra: Mathematical Structure and Dynamical Schrödinger Symmetries*, Springer, 2012.

[6] P. C. Stichel, W. J. Zakrzewski, *Ann. Phys.* **310** (2004) 158; J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Phys. Lett.* **A357** (2006) 1; *ibid.* **B650** (2007) 203.

[7] P. A. Horváthy, L. Martina and P. C. Stichel, *SIGMA* **6** (2010) 060; N. Aizawa, P. S. Isaac and Y. Kimura, Int. J. Math. **23** (2012) 1250118.

[8] N. Aizawa, arXiv:1112.0634 [math-ph].

[9] V. G. Kac, A. K. Raina, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, World Scientific, 1987.

[10] N. Aizawa, P. S. Isaac, *J. Phys. A:Math. Theor.* **44** (2011) 035401.

[11] A. Kirillov, *Lectures on the orbit method*, AMS, 2004.

[12] E. Witten. *Comm. Math. Phys.* **114** (1988) 1.

[13] I. Bakas, *Nucl. Phys.* **B302** (1988) 189.